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TECHNICAL MEMORANDUM 312-409

March 24, 1964

TO: Distribution

FROM: J.D. Anderson

SUBJECT: Theoretical Basis of the JPL Orbit Determination Program

I. Introduction

This document is intended to serve as a supplement to TM 33-168, "The Orbit Determination Program of The Jet Propulsion Laboratory", by R.H. Hudson, M.W. Nead, and M.R. Warner. It attempts to provide a more detailed description of the theoretical basis of the orbit determination program (ODP) than is given in TM 33-168, but it is not strictly necessary to read the material presented here in order to achieve an ability to operate the program. However if one is interested in the derivation of formulas used by the ODP or if information on the basis of particular computational techniques is desired, then this document can be of some use.

In its present form, that is as a Section 312 Technical Memorandum, this document is available to JPL personnel only. Eventually we intend to release it along with TM 33-168 as a Technical Report for wider distribution. Therefore any suggestions pertaining to additions to the content or corrections to erroneous material would be appreciated by the author. Unfortunately the notation of TM 33-168 and this memorandum are not the same, although when the two documents are released in the Technical Report form the notation used here will prevail. The notation used has been described in TM 312-407, "The Standard Nomenclature for Orbit Determination",

Rpt 28950

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by J.D. Anderson, P.R. Peabody, and M.R. Warner, and as such will probably be included in the proposed Technical Report as a table of nomenclature.

II. Statistical Model

The ODP uses a weighted least squares parameter estimation procedure with the refinement that a-priori information on the parameters along with their statistics exert an influence on the estimate. In this form the procedure is often called Bayes estimation and it provides considerable flexibility in that the data from a particular experiment can be partitioned into blocks so that the estimation of parameters for a current block of data can be combined with all previous data blocks through the introduction of the a-priori information. Thus for problems in space flight it is possible to perform the orbit determination in a series of phases. For example on a planetary mission the data from the portion of the flight where the sun dominates the probe might be reduced first with an epoch for the position and velocity parameters (the initial conditions) taken sometime near injection into the heliocentric orbit. Next the parameters and their covariance matrix could be mapped forward to a new epoch at a time shortly before planetary encounter. These mapped quantities would then represent a-priori information for the reduction of data taken when the probe was dominated by the target planet. The advantages in splitting the flight up into these two parts are 1. there is a numerical advantage in that the data is less sensitive to variations in the initial conditions taken at an epoch not too far removed from the times of observation and 2. by separating the problem into two parts, the heliocentric and planet centered data can be analized more or less independently, thus giving the analyst considerably more flexibility in the post flight reduction of the data.

In the formulation of the parameter statistics, an attempt is made to provide a fairly realistic parameter covariance matrix which will depend on the a-priori parameter covariance matrix and on the assumptions as to the statistical model for the data. Also the systematic errors introduced by leaving out important parameters in the solution for the parameter estimate are included in a statistical sense.

A. Estimation Formula

Although summation notation is used in the definition of the normal equations in Section II B of TM 33-168, it is easier to justify the use of various expressions in that section with matrix notation instead. The summation notation as given is indicative of the computational technique used in the processing of the data $\left[z(1), z(2), \ldots, z(N)\right]$, namely the matrices J and $\underline{f}$ are construtted sequentially through the single summations. However in matrix notation the weighted sum of squares of the residuals can be written

$$S = (\hat{\underline{z}} - \underline{z})^T W (\hat{\underline{z}} - \underline{z}) \tag{1}$$

where the residulas $(\hat{\underline{z}} - \underline{z})$ are arranged in an N x 1 column matrix and the weighting matrix W is an N x N diagonal matrix.

$$\hat{\underline{z}} - \underline{z} = \begin{pmatrix} \hat{z}(1) - z(1) \\ \hat{z}(2) - z(2) \\ \vdots \\ \hat{z}(N) - z(N) \end{pmatrix} \qquad W = \begin{pmatrix} w(1) & 0 & \ldots & 0 \\ 0 & w(2) & \ldots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \ldots & w(N) \end{pmatrix}$$

The elements of $\hat{\underline{z}}$ are the actual observations while those of $\underline{z}$ represent the computed values as a function of the parameters $\underline{x}$ which are subject to estimation.

$$\underline{z} = \underline{z}(\underline{x}) \tag{2}$$

The equations of condition relate variations in $\underline{z}$ to variations in $\underline{x}$ through the variational matrix A_x.

$$\delta\underline{z} = A_x \; \delta\underline{x} \tag{3}$$

If there are M parameters in $\underline{x}$ then A is an N x M matrix and in terms of the M x 1 matrices $\underline{a}(i)$ defined in Section II B of TM 33-168 it can be written in a partitioned form.

$$A = \begin{pmatrix} \underline{a}^T(1) \\ \hline \underline{a}^T(2) \\ \hline \\ \hline \underline{a}^T(N) \end{pmatrix}$$

The a-priori term which enters in the normal equations arises from a consideration of the a-priori values $\tilde{\underline{x}}$ of the parameters as additional observations with residuals given by $\tilde{\underline{x}} - \underline{x}$. Thus the weighted sum of squares of all the residuals is written

$$Q = (\hat{\underline{z}} - \underline{z})^T \, W(\hat{\underline{z}} - \underline{z}) + (\tilde{\underline{x}} - \underline{x})^T \, \tilde{\Gamma}_x^{-1} \, (\tilde{\underline{x}} - \underline{x}) \tag{4}$$

where $\tilde{\Gamma}_x^{-1}$ is an M x M generally non diagonal weighting matrix for the $\tilde{\underline{x}}$ observations. The inverse matrix $\tilde{\Gamma}_x$ is interpreted as a covariance matrix on the a-priori parameters $\tilde{\underline{x}}$.

Now a minimization of Q defines the parameter estimate $\underline{x}^*$ in the orbit determination program. The first variation in Q is

$$\delta Q = -2\delta\underline{z}^T W(\underline{\hat{z}} - \underline{z}) - 2\delta\underline{x}^T \tilde{\Gamma}_x^{-1} (\underline{\tilde{x}} - \underline{x}) \tag{5}$$

or from Eq. (3)

$$-\tfrac{1}{2}\,\delta Q = \delta\underline{x}^T \left[A_x^T W(\underline{\hat{z}} - \underline{z}) + \tilde{\Gamma}_x^{-1} (\underline{\tilde{x}} - \underline{x})\right] \tag{6}$$

and in order that Q be stationary for arbitrary parameter variations $\delta\underline{x}$ it is necessary that the term in braces be null. Without discussions of uniqueness or maxima versus minima points the parameter estimate $\underline{x}^*$ is defined by

$$A_x^T W\left[\underline{\hat{z}} - \underline{z}(\underline{x}^*)\right] + \tilde{\Gamma}_x^{-1} (\underline{\tilde{x}} - \underline{x}^*) = 0 \tag{7}$$

From a practical viewpoint all that is required of $\underline{x}^*$ is that it provide an absolute minimization of Q. The fact that it will also satisfy Eq. (7) is incidental. However an $\underline{x}^*$ can be found as a solution to these M non linear algebraic equations in the M unknown parameters; numerical methods are used to actually obtain the solution. First of all a function of $\underline{x}$ is defined as a column matrix of the form

$$A_x^T W\left[\underline{\hat{z}} - \underline{z}(\underline{x})\right] + \tilde{\Gamma}_x^{-1} (\underline{\tilde{x}} - \underline{x}) = \underline{f}(\underline{x}) + \tilde{\Gamma}_x^{-1} (\underline{\tilde{x}} - \underline{x}) \tag{8}$$

The solution to Eq. (7) is obtained by the Newton-Raphson procedure which gives $\underline{x}^*(n+1)$, the value of $\underline{x}^*$ at the n+1 st iteration, in terms of $\underline{x}^*(n)$ at the nth iteration. The procedure requires the first variation of equation (8)

$$\delta\left[\underline{f}(\underline{x}) + \tilde{\Gamma}_x^{-1} (\underline{\tilde{x}} - \underline{x})\right] = -A_x^T W\,\delta\underline{z} - \tilde{\Gamma}_x^{-1}\,\delta\underline{x} \tag{9}$$

or with Eq. 3

$$A_x^T W\,\delta\underline{z} + \tilde{\Gamma}_x^{-1}\,\delta\underline{x} = (A_x^T W A_x + \tilde{\Gamma}_x^{-1})\,\delta\underline{x} \tag{10}$$

Actually the matrix A_x is itself a function of $\underline{x}$ and there is therefore an additional variation in Eq. (9). It is also possible to construct situations where the data weights in W depend on $\underline{x}$; they may be functions of $\underline{z}$, the computed data matrix. However these variations are neglected in the formation of Eq. 10 and in fact experience has shown that the convergence to the solution $\underline{x}^*$ is not adversely affected by the neglect of the variations in A_x and W. Therefore, the iteration formula that is used to obtain the weighted least squares estimate of the orbit is given by

$$(J + \tilde{\Gamma}_x^{-1}) \left[\underline{x}^*(n+1) - \underline{x}^*(n)\right] = \underline{f}(n) + \tilde{\Gamma}_x^{-1} \left[\tilde{\underline{x}} - \underline{x}^*(n)\right] \tag{11}$$

where

$$J = A_x^T W A_x \tag{12}$$

$$\underline{f}(n) = A^T W \left\{ \hat{\underline{z}} - \underline{z} \left[\underline{x}(n)\right] \right\} \tag{13}$$

The matrix A is evaluated for $\underline{x} = \underline{x}(n)$ in the above definitions so far as the program is concerned. Actually this is not always necessary to achieve convergence.

B. Parameter Covariance Matrix

To obtain an expression Γ_x for the covariance matrix on the parameter estimate $\underline{x}^*$ a linearity assumption is introduced. Suppose that the actual values $\underline{x}$ for the parameters were inserted in the iteration formula (12). Then an estimate of the parameters would be computed that would differ from the actual parameters by $\underline{x}^* - \underline{x}$ and in fact the linear expression for this difference is given simply by Eq. (11) above.

$$(J + \tilde{\Gamma}_x^{-1}) (\underline{x}^* - \underline{x}) = \underline{f} + \tilde{\Gamma}_x^{-1} (\tilde{\underline{x}} - \underline{x}) \tag{14}$$

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Now a covariance matrix on a set of random variables $\underline{x}$ can be defined by $\Gamma_x = \overline{\partial \underline{x} \, \partial \underline{x}^T}$ where $\partial \underline{x}$ is the error on $\underline{x}$ and the bar over the quantity indicates its expected value. For example $\overline{\partial x} = 0$ if the error in $\underline{x}$ is distributed about a zero mean. The covariance matrix on the parameter estimate $\underline{x}^*$ is therefore given by

$$\Gamma_x = \overline{(\underline{x}^* - \underline{x}) \, (\underline{x}^* - \underline{x})^T} \tag{15}$$

It may be somewhat confusing to take an ensemble average over $(\underline{x}^* - \underline{x}) \, (\underline{x}^* - \underline{x})^T$ when only one sample of $\underline{x}^*$ is available. However Eq. (15) expresses only a theoretical averaging of the error $\underline{x}^* - \underline{x}$ and thus the matrix Γ_x represents a model of the covariance matrix that could be computed empirically if the space flight in question could be performed many times under exactly the same conditions. The expression for Γ_x is obtained by substituting Eq. 14 in Eq. 15.

$$(J + \tilde{\Gamma}_x^{-1}) \, \Gamma_x \, (J + \tilde{\Gamma}_x^{-1}) = \overline{\underline{f}\underline{f}^T} + \tilde{\Gamma}_x^{-1} \, \overline{(\tilde{\underline{x}} - \underline{x}) \, (\tilde{\underline{x}} - \underline{x})^T} \quad \tilde{\Gamma}_x^{-1} \tag{16}$$

The expected value of $\underline{f} \, (\tilde{\underline{x}} - \underline{x})^T$ and $(\tilde{\underline{x}} - \underline{x}) \, \underline{f}^T$ is set equal to zero which implies that there is no correlation between the errors in the a-priori values of the parameters and the data errors. From equation (11) it is clear that $\left[J + \tilde{\Gamma}_x^{-1} \right] = \left[J + \tilde{\Gamma}_x^{-1} \right]^T$ because covariance matrices are always symmetrical. That is

$$(J + \tilde{\Gamma}_x^{-1})^T = A_x^T \, W^T \, A_x + (\tilde{\Gamma}_x^{-1})^T = A_x^T \, W \, A_x + \tilde{\Gamma}_x^{-1} = (J + \tilde{\Gamma}_x^{-1}) \tag{17}$$

Note that W is diagonal in the program but in any case it is certainly symmetrical.

Now expressions are needed for $\overline{\underline{f}\underline{f}^T}$ and $\overline{(\tilde{\underline{x}} - \underline{x})\ (\tilde{\underline{x}} - \underline{x})^T}$. By definition

$$\overline{(\tilde{\underline{x}} - \underline{x})\ (\tilde{\underline{x}} - \underline{x})^T} = \tilde{\Gamma}_x \tag{18}$$

Before forming $\overline{\underline{f}\underline{f}^T}$ from Eq. 15 a more general situation is considered by recognizing that certain parameters $\underline{y}$ might be used in the formation of $\underline{z}$, but that they might not be included in the set of parameters $\underline{x}$. In other words the set of all parameters $\underline{q}$ that completely specifies the data $\underline{z}$ is partitioned into the set $\underline{x}$ which is subject to estimation and the set $\underline{y}$ whose elements are set equal to their a-priori values $\tilde{\underline{y}}$ and held fixed. Thus $\underline{f}$ is actually evaluated at $\underline{x}$ and $\tilde{\underline{y}}$

$$\underline{f} = A_x^T W \left[\hat{\underline{z}} - \underline{z}\,(\underline{x},\ \tilde{\underline{y}}\,)\right] \tag{19}$$

However the "true" value of $\underline{z}$ is given by $\underline{z}(\underline{x},\ \underline{y})$ so that again with the assumption of linearity

$$\underline{z}(\underline{x},\ \tilde{\underline{y}}) = \underline{z}(\underline{x},\ \underline{y}) + A_y\ (\tilde{\underline{y}} - \underline{y}) \tag{20}$$

where A_y is defined by the relation $\delta\underline{z} = A_x\ \delta\underline{x} + A_y\ \delta\underline{y}$. Substitute Eq. (20) into Eq. (19) and define the error in the data by $\partial\underline{z} = \hat{\underline{z}} - \underline{z}(\underline{x},\ \underline{y})$. Also the error in the parameters $\tilde{\underline{y}}$ is given by $\partial\tilde{\underline{y}} = \tilde{\underline{y}} - \underline{y}$. Then

$$\underline{f} = A_x^T W\ (\partial\underline{z} - A_y\ \partial\tilde{\underline{y}}) \tag{21}$$

Now the data covariance matrix Γ_z is simply $\overline{\partial\underline{z}\ \partial\underline{z}^T}$ while that for the a-priori parameters $\tilde{\underline{y}}$ is $\tilde{\Gamma}_y = \overline{\partial\tilde{\underline{y}}\ \partial\tilde{\underline{y}}^T}$ and if there is no correlation between the data errors and the errors on $\tilde{\underline{y}}$ the expected value of $\overline{\underline{f}\underline{f}^T}$ is

$$\underline{\overline{ff^T}} = A_x^{\ T} W (\Gamma_z + A_y \tilde{\Gamma}_y A_y^{\ T}) W A_x \tag{22}$$

Up to this point everything has been rigorous within the framework of the stated assumptions. However an approximation is now introduced in the orbit determination program, namely that $W = \Gamma_z^{-1}$. This is valid if the data are not correlated and if the elements of W represent the inverse of the variances on the appropriate data. Any differences between W and Γ_z^{-1} cannot be analyzed easily with the program described here. With the indicated approximation

$$\underline{\overline{ff^T}} = A_x^{\ T} W A_x + A_x^{\ T} W A_y \tilde{\Gamma}_y A_y^{\ T} W A_x \tag{23}$$

Substitute Eq. (18) and (23) in (16) to obtain

$$(J + \tilde{\Gamma}_x^{-1}) \Gamma_x (J + \tilde{\Gamma}_x^{-1}) = (J + \tilde{\Gamma}_x^{-1}) + A_x^{\ T} W A_y \tilde{\Gamma}_y A_y^{\ T} W A_x \tag{24}$$

or finally the expression for the covariance matrix Γ_x is

$$\Gamma_x = (J + \tilde{\Gamma}_x^{-1})^{-1} + (J + \tilde{\Gamma}_x^{-1})^{-1} A_x^{\ T} W A_y \tilde{\Gamma}_y A_y^{\ T} W A_x (J + \tilde{\Gamma}_x^{-1})^{-1} \tag{25}$$

The single summations appearing in the formulas of Section II B occur because W is a diagonal matrix. This can be illustrated by considering the calculation of the matrix J. In general the r, s element of J would be given by

$$j_{rs} = \sum_{\alpha=1}^{N} \sum_{\beta=1}^{N} a_{\alpha r} w_{\alpha\beta} a_{\beta s}$$

where a_{rs} and w_{rs} are the rs elements of A and W respectively. However because W is diagonal

$$w_{\alpha\beta} = \delta_{\alpha\beta} w(\alpha) \tag{27}$$

where $\delta_{\alpha\beta}$ is the Kronecker δ - symbol. Therefore

$$j_{rs} = \sum_{\alpha=1}^{N} a_{\alpha r}\, w(\alpha)\, a_{\alpha s} \tag{28}$$

or in terms of the column matrices $\underline{a}(i) = (a_{i1}, a_{i2}, \ldots, a_{iM})$

$$J = \sum_{i=1}^{N} \underline{a}(i)\, w(i)\, \underline{a}^T(i) \tag{29}$$

which is the form used in the program to accumulate J over perhaps several thousand observations.

C. Mapping of the Covariance Matrix

For purposes of forwarding the epoch from a time t_o to a later time t_i or for the study of orbital uncertainties at a midcourse maneuver time, target encounter time and so forth, it is necessary that the parameter covariance matrix Γ_x at t_o be mapped to a parameter covariance matrix Γ_m at t_i. The parameters $\underline{m}$ at t_i are assumed linearly related to both parameters $\underline{x}$ and $\underline{y}$ at t_o by the relation

$$\delta\underline{m} = M_x\, \delta\underline{x} + M_y\, \delta\underline{y} \tag{30}$$

or in terms of errors in $\underline{m}$, $\underline{x}$ and $\tilde{\underline{y}}$

$$\partial\underline{m} = M_x\, \partial\underline{x} + M_y\, \partial\tilde{\underline{y}} \tag{31}$$

and the covariance matrix Γ_m is simply

$$\Gamma_m = \overline{\partial\underline{m}\, \partial\underline{m}^T} \tag{32}$$

or

$$\Gamma_m = M_x \Gamma_x M_x^T + M_x \overline{\partial \underline{x}\, \partial \underline{\tilde{y}}^T} M_y^T + M_y \overline{\partial \underline{\tilde{y}}\, \partial \underline{x}^T} M_x^T + M_y \tilde{\Gamma}_y M_y^T \tag{33}$$

The cross terms $\overline{\partial \underline{x}\, \partial \underline{\tilde{y}}^T}$ and $\overline{\partial \underline{\tilde{y}}\, \partial \underline{x}^T}$ are found by combining Eqs. (14) and (21).

$$\partial \underline{x} = (J + \tilde{\Gamma}_x^{-1})^{-1} \left[A_x^T W \partial \underline{z} - A_x^T W A_y \partial \underline{\tilde{y}} + \tilde{\Gamma}_x \partial \underline{\tilde{x}} \right] \tag{34}$$

and

$$\overline{\partial \underline{x}\, \partial \underline{\tilde{y}}^T} = - (J + \tilde{\Gamma}_x^{-1})^{-1} A_x^T W A_y \tilde{\Gamma}_y \tag{35}$$

where $\overline{\partial \underline{z}\, \partial \underline{\tilde{z}}^T}$ and $\overline{\partial \underline{\tilde{x}}\, \partial \underline{\tilde{y}}^T}$ are assumed null. The other term $\overline{\partial \underline{\tilde{y}}\, \partial \underline{x}^T}$ is the transpose of Eq. (35)

$$\overline{\partial \underline{\tilde{y}}\, \partial \underline{x}^T} = - \tilde{\Gamma}_y A_y^T W A_x (J + \tilde{\Gamma}_x^{-1})^{-1} \tag{36}$$

III. COMPUTATION OF THE OBSERVABLES

In order to apply the least squares estimation formula it is necessary to form the residuals in all the observed quantities. Thus from the current estimate of the orbital parameters $\underline{x}^*$ a position and velocity ephemeris of the probe is constructed. The problem then is to reduce these ephemeris values to actual observed quantities at the time of observation. In this section the only observables discussed are those obtained from optical and radio telescopes fixed on the surface of the earth.

There are various angular measurements obtained from both optical and radio instruments. In addition radio telescopes can provide measurements of the range between the observer and the object as well as the time rate of change of this range measurement; or perhaps more fundamentally with respect to range rate, the time integrated frequency of a doppler signal is measured. Optical angular observations are referred to the earth's equatorial system. This is because accurate observations are obtained by measuring photographs of the object taken against a star background and the stars used for reference on the plate are given in the star atlases in terms of equatorial coordinates. However for radio telescopes the angular observations are obtained directly by recording the direction in which the antenna is pointed and in this situation the angles obtained will be equatorially referenced if the antenna is polar mounted like an astronomical telescope, but for an azimuth-elevation mount, such as in a theodolite, the angles will be referenced to the observer's horizon system.

A. Range and Range Rate

The basis of the various geometric data types is the topocentric range vector $\underline{\rho}$ from the station to the probe. Let $\underline{r}$ be the geocentric position vector of the probe and $\underline{R}$ be the geocentric position vector of the station.

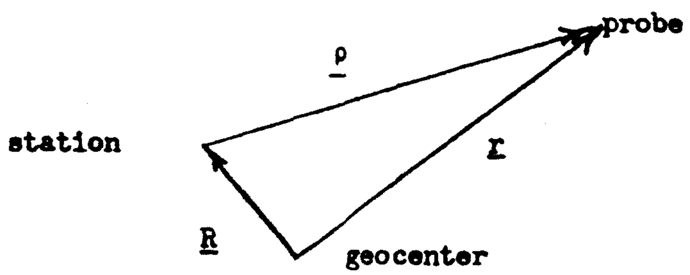

Figure 1

Thus

$$\underline{\rho} = \underline{r} - \underline{R} \tag{37}$$

and the range rate vector $\underline{\dot{\rho}}$ is

$$\underline{\dot{\rho}} = \underline{\dot{r}} - \underline{\dot{R}} \tag{38}$$

The range is the magnitude of $\underline{\rho}$

$$\rho^2 = \underline{\rho} \cdot \underline{\rho} \tag{39}$$

and a differentiation with respect to time gives the range rate $\dot{\rho}$.

$$\rho \, \dot{\rho} = \underline{\rho} \cdot \underline{\dot{\rho}} \tag{40}$$

B. Angles

When angles are used as data the particular coordinate system used to specify $\underline{\rho}$ becomes important. First of all consider the equator of the earth as the reference plane and let the z axis point to the north celestial pole. In addition let x point to the vernal equinox γ and let y complete the right handed cartesian system of coordinates.

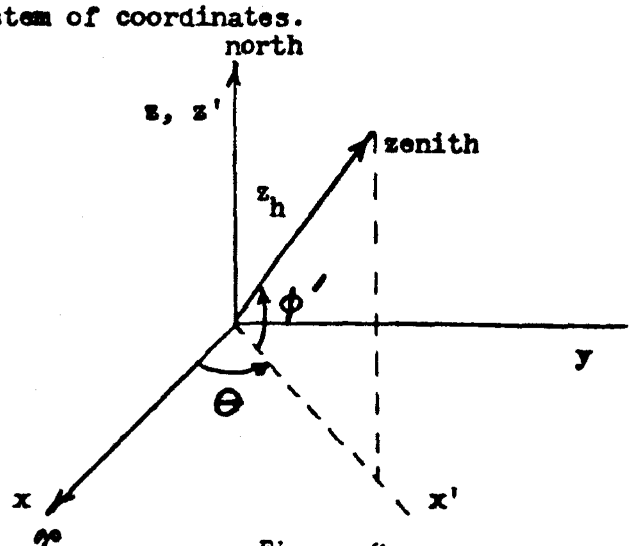

Figure 2

Now a second topocentric set of coordinates is defined such that z_h points to the zenith of the station, x_h is in the plane of the meridian of the station, in other words the plane defined by z and z_h, and again y_h completes the right handed system.

In order to simplify the transformation between (x, y, z) and (x_h, y_h, z_h) the x_h axis is directed towards the south. Thus only two rotations are required for the transformation. First rotate the x axis towards y about z through the angle θ to obtain x', y' and z' = z.

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad (41)$$

Now rotate z' toward x' about y' through the angle ($\pi/2 - \phi'$) to obtain x_h, $y_h = y'$ and z_h.

$$\begin{pmatrix} x_h \\ y_h \\ z_h \end{pmatrix} = \begin{pmatrix} \sin\phi' & 0 & -\cos\phi' \\ 0 & 1 & 0 \\ \cos\phi' & 0 & \sin\phi' \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \quad (42)$$

The complete transformation is obtained by multiplying the two rotation matrices.

$$\begin{pmatrix} x_h \\ y_h \\ z_h \end{pmatrix} = \begin{pmatrix} \sin\phi'\cos\theta & \sin\phi'\sin\theta & -\cos\phi' \\ -\sin\theta & \cos\theta & 0 \\ \cos\phi'\cos\theta & \cos\phi'\sin\theta & \sin\phi' \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad (43)$$

The angle θ is the local sidereal time at the station and it is obtained by adding the station longitude λ to the Greenwich sidereal time θ or Greenwich hour angle (GHA) which is computed by the formulas given by D.B. Holdridge in TR No. 32-223, "Space Trajectories Program for the IBM 7090 Computer".

$$\theta = \theta_G + \lambda \quad (44)$$

Now the angle observations obviously can be interpreted in terms of the direction of the probe or equivalently in terms of the unit vector $\underline{L}$ in the direction of the range vector.

$$\underline{L} = \frac{\underline{\rho}}{\rho} \tag{45}$$

It is understood that the coordinate system which defines $\underline{L}$ is the equatorial system, the coordinates of the probe and station are given in geocentric equatorial coordinates. Therefore $\underline{\rho}$ is computed by Eq. 37, ρ by Eq. 39 and $\underline{L} = (L_x, L_y, L_z)$ by Eq. 45.

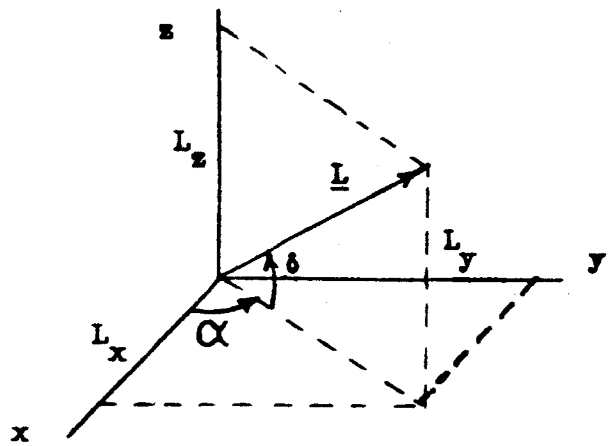

Figure 3

The above diagram shows the unit vector $\underline{L}$ as resolved into its three components in the equatorial system and defines the topocentric right ascension α and declination δ in the conventional manner.

Thus

$$\left.\begin{aligned} L_x &= \cos\delta \cos\alpha \\ L_y &= \cos\delta \sin\alpha \\ L_z &\quad \sin\delta \end{aligned}\right\} \tag{46}$$

and α and δ can be computed. In the ODP the declination is obtained by

$$\delta = \sin^{-1} L_z \qquad (-90^\circ \leqq \delta \leqq 90^\circ) \tag{47}$$

and the right ascension by

$$\alpha = \tan^{-1} \frac{L_y}{L_x} \tag{48}$$

or actually with $\underline{\rho} = (\xi, \eta, \zeta)$, the right ascension is computed by the following

$$\alpha = \tan^{-1} \frac{\eta}{\xi} \qquad (0 \leqq \alpha < 360^\circ) \tag{49}$$

The quadrant of α is determined by inspecting the signs of L_x and L_y. (Note that $\cos\delta \geqq 0$) When using angles from polar mounted radio telescopes the hour angle H is observed rather than the right ascension. It is defined by the relation

$$H = \theta - \alpha \tag{50}$$

The second set of coordinates (x_h, y_h, z_h) defined above are useful in computing the elevation angle γ and azimuth σ which represent angular data from a radio telescope whose mounting is referenced to the station's horizon rather than the equator of the earth.

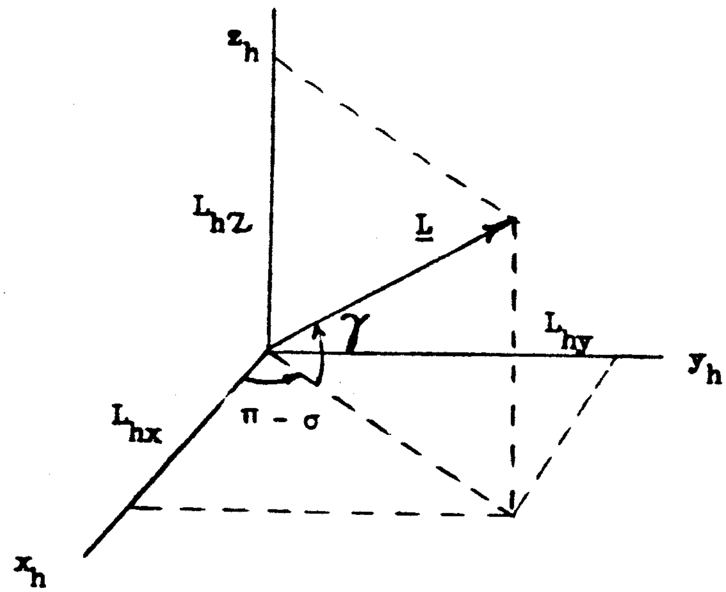

Figure 4

The azimuth angle as indicated in the diagram is measured from the north toward the east through 360°. Thus

$$\left.\begin{aligned} L_{hx} &= -\cos \gamma \cos \sigma \\ L_{hy} &= \cos \gamma \sin \sigma \\ L_{hz} &= \sin \gamma \end{aligned}\right\} \qquad (51)$$

The components L_{hx}, L_{hy}, L_{hz} can be computed by the transformation of Eq. 43, and σ and γ follow just as α and δ were obtained from L_x, L_y, and L_z.

$$L_{hx} = L_x \sin \phi' \cos \theta + L_y \sin \phi' \sin \theta - L_z \cos \phi' \qquad (52)$$

$$L_{hy} = -L_x \sin \theta + L_y \cos \theta \qquad (53)$$

$$L_{hz} = L_x \cos \phi' \cos \theta + L_y \cos \phi' \sin \theta + L_z \sin \phi' \qquad (54)$$

Eq. 54 can be expressed in terms of station coordinates X, Y, and Z.

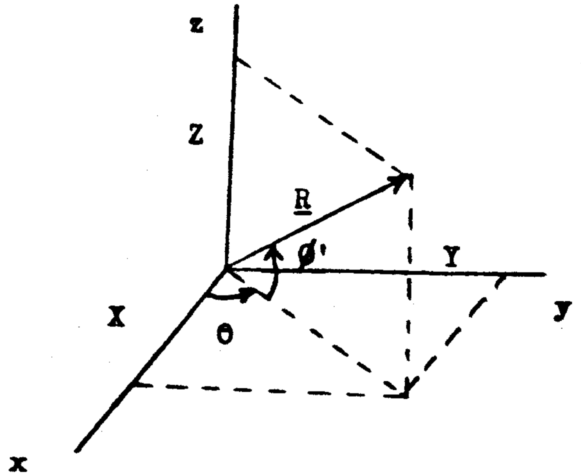

Figure 5

$$\left.\begin{aligned} X &= R \cos \phi' \cos \theta \\ Y &= R \cos \phi' \sin \theta \\ Z &= R \sin \phi' \end{aligned}\right\} \qquad (55)$$

and the alternative form of Eq. 18 is

$$L_{hz} = \frac{1}{R} (L_x X + L_y Y + L_z Z) \qquad (56)$$

The fact that sin γ is equal to the right hand side of Eq. 56 can be seen immediately by considering the meaning of the scaler product $\underline{L} \cdot \underline{R}$. Note that the vector $\underline{R}$ points to the geocentric zenith.

$$\underline{L} \cdot \underline{R} = R \cos (\underline{L}, \underline{R}) = R \cos \left(\frac{\pi}{2} - \gamma\right)$$

or (57)

$$\underline{L} \cdot \underline{R} = R \sin \gamma$$

C. Time of Occultation

The time of occultation T_{occ} when the probe disappears behind a body of radius a_p is also a data type and must be computed by an iterative technique. Consider the general situation where the probe is located at any point in space.

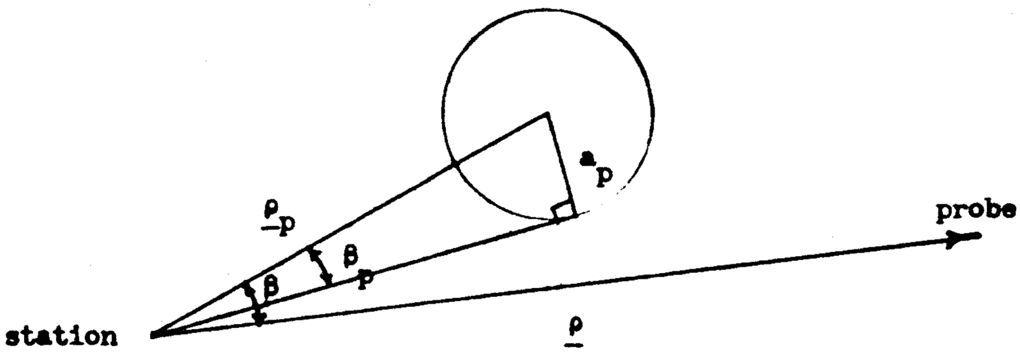

Figure 6

The angle $(\beta - \beta_p)$ represents the angular distance of the probe from the assumed spherical body whose topocentric position is given by $\underline{\rho}_p$. Now this angle is a function of time because of the motion of the probe, the body and the station.

$$\beta - \beta_p = f(t) \tag{58}$$

However the particular value of time where $f(t) = 0$ is the required time of occultation, and therefore T_{occ} can be found by the Newton - Raphson iteration formula

$$T_{occ}(n+1) = T_{occ}(n) - \frac{f\left[T_{occ}(n)\right]}{\dot{f}\left[T_{occ}(n)\right]} \tag{59}$$

or

$$T_{occ}(n+1) = T_{occ}(n) - \frac{\beta(n) - \beta_p(n)}{\dot{\beta}(n) - \dot{\beta}_p(n)} \tag{60}$$

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where $\beta(n)$, $\beta_p(n)$, $\dot{\beta}(n)$ and $\dot{\beta}_p(n)$ are all evaluated at $T_{occ}(n)$ from the n^{th} iteration. From figure 5

$$\sin \beta_p = \frac{a_p}{\rho_p} \tag{61}$$

where

$$\rho_p^{\ 2} = \underline{\rho}_p \cdot \underline{\rho}_p \tag{62}$$

Also

$$\cos \beta = \frac{\underline{\rho} \cdot \underline{\rho}_p}{\rho \rho_p} \tag{63}$$

The vector $\underline{\rho}_p$ is evaluated by subtracting the geocentric position of the station from that of the body. Now the time derivatives of β and β_p are required. Differentiate Eq. (61) and obtain

$$\dot{\beta}_p = -\frac{\dot{\rho}_p}{\rho_p} \tan \beta_p = -\frac{\dot{\rho}_p}{\rho_p} \frac{a_p}{\sqrt{\rho_p^{\ 2} - a_p^{\ 2}}} \tag{64}$$

where

$$\rho_p \dot{\rho}_p = \underline{\rho}_p \cdot \underline{\dot{\rho}}_p \tag{65}$$

Similarly $\dot{\beta}$ is obtained by differentiating Eq. (63)

$$\rho\rho_p \sin \beta \dot{\beta} = (\dot{\rho}\rho_p + \rho\dot{\rho}_p) \cos \beta - (\underline{\rho} \cdot \underline{\dot{\rho}}_p + \underline{\dot{\rho}} \cdot \underline{\rho}_p) \tag{66}$$

Of course light time corrections must be applied in computing both $(\underline{\rho}, \underline{\dot{\rho}})$ and $(\underline{\rho}_p, \underline{\dot{\rho}}_p)$; the time T_{occ} is recorded by the station as the time when the signal from the probe either ceases or commences again after a silence. Clearly from

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the diagram there are two solutions for T_{occ}, one when the probe is occulted and another when it comes out from behind the body. Thus the initial guess to T_{occ} determines which solution the formula (24) will give. The procedure is to use the actual observed values of the two times as first guesses. It has been found that they are sufficiently close to the solutions so that no difficulty is encountered in separating the two values of T_{occ}.

Before proceeding to the observables which concern radio measurements only, we point out that there is an entire area in the field of classical orbit determination that deals with the computation of angle residuals. This method, often called differential representation, computes the residuals directly without the necessity of computing the angles themselves. The subject is treated in the literature and is extremely powerful when only angle observations are available. However, this is seldom the case in space flight problems where range and range rate measurements are used, and in the ODP the angle residuals are obtained by differencing the observed minus the computed values.

D. Doppler Data

As has been stated earlier the measurement of range rate is not fundamental to radio telescopes. Instead, the number N of cycles that an electromagnetic wave has completed in some time interval is counted and the results are sometimes interpreted as measurements of range rate. In order that residuals can be formed in the actual observable, formulas are derived in this section that relate a received frequency ν_{ob} to a known transmitted frequency ν_{tr}. The received frequency will in general vary with the time so that the cycle count number N or actually the cycle count divided by the count times τ is expressed as an integral.

$$\frac{N}{\tau} = f = \frac{1}{\tau} \int_{t_{ob}-\tau}^{t_{ob}} F(t)\, dt \tag{67}$$

In practice this integral is evaluated numerically. The quantity f is the actual doppler data type and is in units of cycles per second. The function F(t) depends on the ratio $\frac{\nu_{ob}}{\nu_{tr}}$. Also, there are usually various additive and multiplicative constants inserted under the integral in equation (67) to account for the peculiarities of the electronic cycle count device used at the station.

There are many possibilities for measuring a doppler frequency ν_{ob}, especially when two or more observatories are involved in transmitting and receiving radio signals. However, only two possibilities are considered here because practically any other possibility can be derived from them. The two considered ones are one way doppler, where a signal is sent from one point and received at another, and two way doppler, where a signal is sent from one point reflected off of a target and received at a second point. One way doppler is applicable when the probe is equipped with a built in transmitter.

1. Cycle Count Computation

For the data type called coherent three way doppler in the ODP the function F(t) is given by

$$F(t) = \omega_3 + \omega_4\, \nu_{tr} \left(1 - \frac{\nu_{ob}}{k\nu_{tr}}\right) \tag{68}$$

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where ω_3 and ω_4 are given constants dependent on the mechanization of the radio equipment, k is the ratio of the frequency transmitted by the probe to the frequency that it receives from the ground transmitter (k = 1 for reflection), ν_{tr} is the constant frequency of the transmitted signal and $\nu_{ob}/k\nu_{tr}$ is the ratio of frequencies derived in this section. The formula for F(t) in the one way case can be found in TM 33-168 and involves the ratio of the frequency transmitted by the probe to the frequency received at the station.

Once F(t) has been specified for the particular data type of interest, the integral of Eq. (67) can be evaluated to obtain the computed value of the doppler observable. Rather than use standard quadrature formulas for its evaluation, the function F(t) is expanded in a power series about the mid point t_m of the count interval $(t_{ob} - \tau \leq t \leq t_{ob})$ and the result is integrated term by term. Thus the evaluation of F(t) at several points within the interval is avoided in favor of a single evaluation at t_m.

$$F(t) = F(t_m) + \dot{F}(t_m)(t - t_m) + \frac{1}{2}\ddot{F}(t_m)(t - t_m)^2 + \ldots \tag{69}$$

and the integral of Eq. 67 becomes

$$\int_{t_{ob}-\tau}^{t_{ob}} F(t)\,dt = \int_{t_m - \frac{1}{2}\tau}^{t_m + \frac{1}{2}\tau} \left[F(t_m) + \dot{F}(t_m)(t-t_m) + \frac{1}{2}\ddot{F}(t_m)(t-t_m)^2 + \ldots\right] dt \tag{70}$$

or

$$\int_{t_m - \frac{1}{2}\tau}^{t_m + \frac{1}{2}\tau} F(t)\,dt = \tau F(t_m) + \frac{\tau^3}{24}\ddot{F}(t_m) + \ldots \tag{71}$$

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Thus for coherent three way doppler Eqs. (67), (68) and (71) are combined to give

$$f_{c3} = \omega_3 + \omega_4 \nu_{tr} \left(1 - \frac{\nu_{ob}}{k\nu_{tr}}\right) - \frac{\omega_4 \nu_{tr} \tau^2}{24} \left(\frac{\nu_{ob}}{k\nu_{tr}}\right)^{\bullet\bullet} \tag{72}$$

In the derivation of $\nu_{ob}/_{k\,\nu_{tr}}$ terms of order higher than $1/_{c}2$ are neglected. However in forming the second time derivative of $\nu_{ob}/_{k\,\nu_{tr}}$ terms of order higher than $1/_{c}$ are neglected in the ODP.

2. Doppler Frequency Shift

The doppler shift that occurs when a ground transmitter sends a signal to the probe which in turn sends the signal back to the ground where it is received can be derived by taking the center of the earth as the origin of coordinates. For the case where the transmitter and receiver are not both earth based it becomes necessary to take the origin at some practically inertial point such as the center of the sun. However the development is very similar to the one given here.

Consider first the transmission from the probe at P to the station at S and denote the geocentric origin by O.

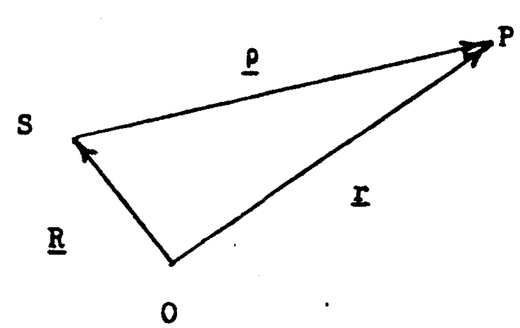

Figure 7

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From the theory of relativity the proper time $d\tau_p$ associated with the source at P is given by

$$d\tau_p^{\ 2} = \left(1 - \frac{v_p^{\ 2}}{c^2} - \frac{2\phi_p}{c^2}\right) dt_p^{\ 2} \tag{73}$$

where v_p is the speed of P with respect to the origin O, ϕ_p is the gravitational potential at P and again c is the constant speed of propagation of the signal. The formula (1) states that if an observer at rest with respect to O measures a time differential dt_p then the corresponding time differential measured at P is $d\tau_p$. Actually the gravitational term in the expression for $d\tau_p^{\ 2}$ is only approximate, although for the Schwarzschild metric it is accurate to order $1/c^2$. In the following the doppler equation itself will be developed only to order $1/c^2$ so formula (73) is satisfactory for the present. Analogous to P the proper time for the station S is

$$d\tau_{ob}^{\ 2} = \left(1 - \frac{v_{ob}^{\ 2}}{c^2} - \frac{2\phi_{ob}}{c^2}\right) dt_{ob}^{\ 2} \tag{74}$$

Now the transmitted frequency ν_{PT} at P is simply the inverse of the time increment $\Delta\tau_p$ between successive peaks of the electromagnetic wave. Similarly the received frequency at S is the inverse of $\Delta\tau_{ob}$. From this point on all time increments will be approximated by the corresponding differential expressions. It should be noted that through powers in $1/c^2$

$$d\tau = \left(1 - \frac{v^2}{2c^2} - \frac{\phi}{c^2}\right) dt \tag{75}$$

so that

$$\Delta\tau = \left(1 - \frac{v^2}{2c^2} - \frac{\Phi}{c^2}\right)\Delta t - \frac{1}{2c^2} \ (v\ \dot{v} + \dot{\Phi})\,\Delta t^2 + \ldots \tag{76}$$

or with

$$\Delta\tau \approx \frac{1}{\nu}\left(1 - \frac{v^2}{2c^2} - \frac{\Phi}{c^2}\right)\left(1 - \frac{v\ \dot{v} + \dot{\Phi}}{2\nu c^2}\right) \tag{77}$$

Therefore the second order term $(v\ \dot{v} + \dot{\Phi})/2\nu c^2$ is neglected. Certainly this term is small with respect to unity even for radio frequencies.

The expression for the ratio of received to transmitted frequencies is

$$\frac{\nu_{ob}}{\nu_p} = \frac{\Delta\tau_p}{\Delta\tau_{ob}} = \frac{d\tau_p}{d\tau_{ob}} \tag{78}$$

and from Eqs. 1 and 2, the ratio of frequencies to order $1/c^2$ is

$$\frac{\nu_{ob}}{\nu_p} = \left[1 + \frac{1}{2c^2} \ ({v_{ob}}^2 - {v_p}^2) + \frac{1}{c^2} \ (\Phi_{ob} - \Phi_p)\right] \frac{dt_p}{dt_{ob}} \tag{79}$$

The derivative dt_p/dt_{ob} can be obtained by considering the finite propagation of the signal over the distance from P to S.

$$t_p = t_{ob} - \frac{\rho(ob)}{c} \tag{80}$$

where $\rho(ob)$ is the magnitude of the vector

$$\underline{\rho}(ob) = \underline{r}(t_p) - \underline{R}(t_{ob};rec) \tag{81}$$

The position of the probe $\underline{r}(t)$ is evaluated at the time t_p when it sends the signal and the position of the receiver $\underline{R}(t;rec)$ is evaluated at the time of reception t_{ob}.

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Differentiate Eq. 80

$$dt_p = dt_{ob} - \frac{d\rho(ob)}{c} \tag{82}$$

By definition

$$\rho^2(ob) = \underline{\rho}(ob) \cdot \underline{\rho}(ob) \tag{83}$$

Therefore

$$\rho(ob)\, d\rho(ob) = \underline{\rho}(ob) \cdot d\underline{\rho}(ob) \tag{84}$$

and from Eq. (81)

$$d\underline{\rho}(ob) = \underline{\dot{r}}(t_p)\, dt_p - \underline{\dot{R}}(t_{ob};rec)\, dt_{ob} \tag{85}$$

Combine Eqs. (84) and (85) and substitute the result into Eq. (82) to obtain the required ratio

$$\left[1 + \frac{\underline{L}(ob)\cdot\underline{\dot{r}}(t_p)}{c}\right] \frac{dt_p}{dt_{ob}} = 1 + \frac{\underline{L}(ob)\cdot\underline{\dot{R}}(t_{ob};rec)}{c} \tag{86}$$

where $\underline{L}(ob)$ is the unit vector $\underline{\rho}(ob)/\rho(ob)$. Expand Eq. 86 to order $1/c^2$ to obtain

$$\frac{dt_p}{dt_{ob}} = 1 - \frac{\dot{\rho}(ob)}{c} + \frac{\dot{\rho}(ob)}{c^2}\ \underline{L}(ob)\cdot\underline{\dot{r}}(t_p) \tag{87}$$

The range rate $\dot{\rho}(ob)$ is defined by

$$\dot{\rho}(ob) = \underline{L}(ob)\cdot \left[\underline{\dot{r}}(t_p) - \underline{\dot{R}}(t_{ob};\ rec)\right] \tag{88}$$

Now substitute Eq. (87) into Eq. (79) and drop terms of order higher than $1/c^2$.

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$$\frac{\nu_{ob}}{\nu_p} = 1 - \frac{\dot{\rho}(ob)}{c} + \frac{1}{c^2} \left[\dot{\rho}(ob)\, \underline{L}(ob)\cdot\dot{\underline{r}}(t_p) + \tfrac{1}{2}\left(v_{ob}^{\;2} - v_p^{\;2}\right) + \left(\phi_{ob} - \phi_p\right)\right] \tag{89}$$

Now consider the transmission from a station with position $\underline{R}(t_{tr};\, tr)$ to the probe at the same position $\underline{r}(t_p)$ as before. The proper time at the transmitter is

$$d\tau_{tr}^{\;2} = \left(1 - \frac{v_{tr}^{\;2}}{c^2} - \frac{2\phi_{tr}}{c^2}\right) dt_{tr}^{\;2} \tag{90}$$

and the proper time at the probe is given by Eq. (73). Thus the ratio of the frequency ν_R received at the probe to that transmitted ν_{tr} by the station is

$$\frac{\nu_R}{\nu_{tr}} = \left[1 + \frac{1}{2c^2}\left(v_p^{\;2} - v_{tr}^{\;2}\right) + \frac{1}{c^2}\left(\phi_p - \phi_{tr}\right)\right] \frac{dt_{tr}}{dt_p} \tag{91}$$

The relation between t_{tr} and t_p is

$$t_{tr} = t_p - \frac{\rho(tr)}{c} \tag{92}$$

where $\rho(tr)$ is the magnitude of the vector

$$\underline{\rho}(tr) = \underline{r}(t_p) - \underline{R}(t_{tr};tr) \tag{93}$$

Note that in general the transmitter and receiver are not at the same point on the earth's surface so that $\underline{R}(t;rec) \neq \underline{R}(t;\, tr)$ even when the two positions are evaluated at the same time.

The ratio $^{dt_{tr}}/_{dt_p}$ is found by differentiating Eq. (92) and by expanding to order $^1/_{c^2}$ as was done for the probe-receiver transmission.

$$\frac{dt_{tr}}{dt_p} = 1 - \frac{\dot{\rho}(tr)}{c} - \frac{\dot{\rho}(tr)}{c^2}\, \underline{L}(tr) \cdot \dot{\underline{R}}(t_{tr};tr) \tag{94}$$

Therefore the ratio $^{\nu_R}/_{\nu_{tr}}$ is to order $^1/_{c^2}$

$$\frac{\nu_R}{\nu_{tr}} = 1 - \frac{\dot{\rho}(tr)}{c} - \frac{1}{c^2}\left[\dot{\rho}(tr)\, \underline{L}(tr) \cdot \dot{\underline{R}}(t_{tr};tr) - \tfrac{1}{2}\left(v_p^{\,2} - v_{tr}^{\,2}\right) - \left(\phi_p - \phi_{tr}\right)\right] \tag{95}$$

where $\dot{\rho}(tr)$ is defined by

$$\dot{\rho}(tr) = \underline{L}(tr) \cdot \left[\dot{\underline{r}}(t_p) - \dot{\underline{R}}(t_{tr};\, tr)\right] \tag{96}$$

The complete doppler shift over the path from the transmitter to the probe and then to the receiver is found by combining Eqs. (89) and (95). The frequency ν_p transmitted by the probe is assumed proportional to the frequency ν_R received by the probe.

$$\nu_p = k\ \nu_R \tag{97}$$

The constant k is unity for a reflection of the signal.

Therefore

$$\frac{\nu_{ob}}{\nu_{tr}} = k\, \frac{\nu_{ob}}{\nu_p}\ \frac{\nu_R}{\nu_{tr}} \tag{98}$$

or to order $1/c^2$

$$\frac{1}{k}\,\frac{\nu_{ob}}{\nu_{tr}} = 1 - \frac{\dot{\rho}(tr) + \dot{\rho}(ob)}{c} + \frac{1}{c^2}\left\{ \dot{\rho}(ob)\,\dot{\rho}(tr) + \dot{\rho}(ob)\,\underline{L}(ob)\cdot\dot{\underline{r}}(t_p) - \dot{\rho}(tr)\,\underline{L}(tr)\cdot\dot{\underline{R}}(t_{tr};tr) + \tfrac{1}{2}\left(v_{ob}^{\,2} - v_{tr}^{\,2}\right) + \left(\Phi_{ob} - \Phi_{tr}\right)\right\} \tag{99}$$

The $1/c$ term in Eq. (27) is simple enough, but the quantities in the $1/c^2$ term can be expressed in a number of different forms. It is interesting that the relativistic terms, which were introduced in the expression for proper time, are completely independent of the state of the probe at any time. Thus the gravitational doppler shift occurs only because transmission and reception occur at slightly different gravitational potentials but the gravitational potential for the probe has no effect. Since both receiver and transmitter are on the surface of the earth the difference in $\Phi_{ob} - \Phi_{tr}$ occurs because the two points are at slightly different positions in the luni-solar gravitational field and because of differences in the geocentric radii of the two stations. Both of these effects are neglected in the final doppler formula used in the ODP and thus $\Phi_{ob} - \Phi_{tr}$ is set equal to zero. The difference in the squared speeds of the two stations is retained however so that

$$v_{ob}^{\,2} - v_{tr}^{\,2} = \dot{\underline{R}}(t_{ob};rec)\cdot\dot{\underline{R}}(t_{ob};rec) - \dot{\underline{R}}(t_{tr};tr)\cdot\dot{\underline{R}}(t_{tr};tr) \tag{100}$$

Of the remaining terms, $\dot{\rho}(ob)\ \dot{\rho}(tr)$ is retained as is. However the other two terms are expressed in a different form by substituting $\dot{\underline{\rho}}(ob) + \dot{\underline{R}}\ (tob;rec)$ for $\dot{\underline{r}}(tp)$ and by setting the scaler products $\underline{R}(tob;\ rec) \cdot \dot{\underline{R}}(tob;\ rec)$ and $\underline{R}(t_{tr};\ tr) \cdot \dot{\underline{R}}(t_{tr};\ tr)$ equal to zero. In other words the time rate of change of the geocentric radii of the two stations is assumed zero. Thus

$$\dot{\rho}(ob)\ \underline{L}(ob) \cdot \dot{\underline{r}}(tp) - \dot{\rho}(tr)\ \underline{L}(tr) \cdot \dot{\underline{R}}(t_{tr};\ tr)$$

$$= \dot{\rho}^2(ob) + \underline{r}(tp) \cdot \left[\frac{\dot{\rho}(ob)}{\rho(ob)}\ \dot{\underline{R}}(tob;\ rec) - \frac{\dot{\rho}(tr)}{\rho(tr)}\ \dot{\underline{R}}\ (t_{tr};tr) \right] \tag{101}$$

and the final form of the ratio ${}^{\nu_{ob}}/_{\nu_{tr}}$ as used in the ODP is

$$\frac{1}{k}\ \frac{\nu_{ob}}{\nu_{tr}} = 1 - \frac{\dot{\rho}(tr) + \dot{\rho}(ob)}{c} + \frac{1}{c^2}\ \left[\dot{\rho}(ob)\ \dot{\rho}(tr) + \dot{\rho}^2(ob) + H \right] \tag{102}$$

where

$$H = \underline{r}(tp) \cdot \left[\frac{\dot{\rho}(ob)}{\rho(ob)}\ \dot{\underline{R}}(tob;\ rec) - \frac{\dot{\rho}(tr)}{\rho(tr)}\ \dot{\underline{R}}(t_{tr};\ tr) \right] + \tfrac{1}{2} \left[\dot{\underline{R}}(tob;rec) \cdot \dot{\underline{R}}(tob;\ rec) - \dot{\underline{R}}(t_{tr};tr) \cdot \dot{\underline{R}}(t_{tr};\ tr) \right] \tag{103}$$

IV CORRECTIONS TO THE OBSERVABLES

The corrections to radio observations are handled somewhat differently in the ODP than the classical astronomical corrections to optical angles only. For example precession and nutation are accounted for by requiring that the calculated position and velocity ephemeris of the probe be in true coordinates. True equatorial coordinates are defined such that the x axis is directed to the actual vernal equinox at the time in question. The z axis is directed toward the actual north pole and y completes the right hand system. Because the earth is not a perfect sphere, other bodies in the solar system produce torques on it that cause the true x, y, z coordinate system to vary with time. Therefore the published positions of celestial bodies are usually referred to some fixed mean coordinate system at a reference epoch t_o. A rigorous integration of the equations of motion for an observed object requires the introduction of the coordinates of various bodies in the solar system and for this reason, in addition to the reason that the equations of motion take on their simplest form in an inertially fixed coordinate system, the coordinates of the probe are obtained by an integration performed in the coordinates of t_o. It is then necessary to transform to the true coordinates by applying a matrix rotation. The elements of this rotation matrix are given as a function of $t-t_o$ in TR 32-223, pages 66-68 and it is the true coordinates that are available to the ODP.

The subject of paralax, that is the effect of the difference between the position of the station and the origin of the object's position and velocity ephemeris, is completely included in the computation of the observables by requiring that the range and range rate vectors be referrenced to the station.

Time abberration occurs because light, or for that matter any electromagnetic phenomenon, takes a finite length of time to travel from the object to the observer. Thus, although one obtains a measurement at the observation time tob, the object was not at its position corresponding to tob when it sent the signal. Therefore, it is necessary to compute the position at an earlier time of transmission $t_p = tob - \Delta t_p$ so that the observable can be compared with the computed value directly. Unfortunately the time of transmission cannot be determined exactly unless the range ρ defined in equation (39) is known beforehand. Therefore an iterative technique must be applied to obtain Δt_p.

The corrections described above are all accounted for in the computation of the angles, range, range rate and doppler frequency. However there are additional corrections which are applied as increments to the observables. These increments can either be added to the computed values of the observables or to the data themselves, although in the ODP the effects of deflections in the local verticle and refraction are accounted for by correcting the computation of the data.

When processing optical data in the ODP it must be available in the geocentric coordinate system for the true equator and equinox at the observation time.

A. Light Time

The light time correction is concerned with relating three event times so that the range vectors from the transmitter to the probe and from the receiver to the probe can be represented accurately. The time of reception of the electromagnetic signal at the station is t_{ob} while the time at which the probe sends the signal to the receiver is t_p. A third time is required if a ground transmitter is involved in the system. It is the time of transmission t_{tr}. The reception of a signal at the probe and its transmission by the probe to the ground are assumed to be simultaneous events.

The procedure makes use of a time t which is an approximation to t_{pr} and which is defined by

$$t = t_{ob} - \frac{1}{c}\left[r(t) - a_e\right], \qquad (104)$$

where r(t) is the geocentric distance to the probe at time t, a_e is the mean equatorial radius of the earth and c is the speed of light. Now all quantities required at time t_{pr} are computed at t instead and are then linearly corrected by the time increment $\epsilon_t = t_p - t$

For example the position vector of the probe at t_p is given by

$$\underline{r}_p(t_p) = \underline{r}_p(t) + \epsilon_t \, \dot{\underline{r}}_p(t) \tag{105}$$

The rigorous light time correction is

$$\Delta t_p = t_{ob} - t_p \tag{106}$$

while the approximate correction of Eq. (1) is

$$\Delta t = t_{ob} - t = \frac{1}{c} \left[r(t) - a_e \right] \tag{107}$$

Therefore the increment ϵ_t required to correct the quantities evaluated at t is

$$\epsilon_t = \Delta t - \Delta t_p \tag{108}$$

and an expression for Δt_p is needed as a function of quantities evaluated at t. The advantage of introducing t in this way is that when iterating on the light time correction to find $\underline{r}_p(tr)$ say, the probe ephemeris is entered only once at time t.

Now the rigorous correction Δt_p is given by

$$\Delta t_p = \frac{\rho(t_p, t_{ob})}{c} \tag{109}$$

where $\rho(t_p, t_{ob})$ is the distance from the probe at time t_p to the station at time t_{ob}, or $\rho(t_p, t_{ob})$ is the magnitude of the vector

$$\underline{\rho}(t_p, t_{ob}) = \underline{r}_p(t_p) - \underline{R}(t_{ob}) \tag{110}$$

However $\underline{r}_p(t_p)$ is not available except in the form of Eq. (105) and

$$\underline{\rho}(t_p, t_{ob}) = \underline{r}_p(t) - \underline{R}(t_{ob}) + \epsilon_t \, \underline{\dot{r}}_p(t) \tag{111}$$

or with the computable vector

$$\underline{\rho}(t, t_{ob}) = \underline{r}_p(t) - \underline{R}(t_{ob})$$

$$\underline{\rho}(t_p, t_{ob}) = \underline{\rho}(t, t_{ob}) + \epsilon_t \, \underline{\dot{r}}_p(t) \tag{112}$$

and to the first order in ϵ_t the magnitude of the range vector is

$$\rho(t_p, t_{ob}) = \rho(t, t_{ob}) + \epsilon_t \, \frac{\underline{\dot{r}}_p(t) \cdot \underline{\rho}(t, t_{ob})}{\rho(t, t_{ob})} \tag{113}$$

If the term $\underline{\dot{R}} \cdot \underline{\rho}/\rho$ is neglected in forming $\underline{\dot{r}} \cdot \underline{\rho}/\rho = (\underline{\dot{\rho}} + \underline{\dot{R}}) \cdot \underline{\rho}/\rho$ then

$$\rho(t_p, t_{ob}) = \rho(t, t_{ob}) + \epsilon_t \, \dot{\rho}\,(t, t_{ob}) \tag{114}$$

Substitute Eq. 114 into Eq. 109 and make use of Eq. 108 to obtain

$$\Delta t_p = \frac{\rho(t, t_{ob}) + \Delta t \, \dot{\rho}(t, t_{ob})}{c + \dot{\rho}(t, t_{ob})} \tag{115}$$

Thus Eqs. (105), (108), and (115) are repeatedly applied until convergence is achieved. The first guess at ϵ_t is taken as zero. The increment Δt is computed once and for all after t has been obtained from Eq. (104). Of course the observation time t_{ob} is known, although the time of transmission, if applicable, is unknown. It is approximated by the formula

$$t_{tr} = t_{ob} - 2 \, \Delta t_p \tag{116}$$

and the range vector for the transmitter is simply

$$\underline{\rho}(t_p, t_{tr}) = \underline{r}_p(t) - \underline{R}(t_{tr}) + \epsilon_t \, \dot{\underline{r}}_p(t) \tag{117}$$

B. Local Vertical Corrections

Because the observer's zenith is not necessarily incident with the geocentric zenith, the angles computed by section III.B. are not consistant with the observed angles. The geocentric zenith Z' is defined by the intersection with the celestial sphere of the line joining the center of the earth and the observer. On the other hand the observer's zenith Z is defined by the intersection with the sphere of the line defined by the direction of the local vertical, or equivalently by the direction of a plumb bob suspended at the observation site. The difference in Z and Z' naturally seperates into two effects. The largest of these is the correction from the geocentric zenith to the geodetic zenith Z which is defined by the intersection with the celestial sphere of the normal to the spheroid of reference. The second effect is the small corrections to the zenith which account for local gravity anomalies and departures from the reference spheroid. In order to compute the first correction the direction of the geodetic zenith is precisely defined by taking as a reference spheroid an oblate spheroid centered at the center of the earth and oriented such that the minor axis of the spheroid is aligned with the earth's polar axis. Thus an equatorial cross section of the spheroid is circular while a meridianal cross section is elliptical. This implies that the earth's pole, the geodetic zenith and the geocentric zenith all lie in the same meridianal plane.

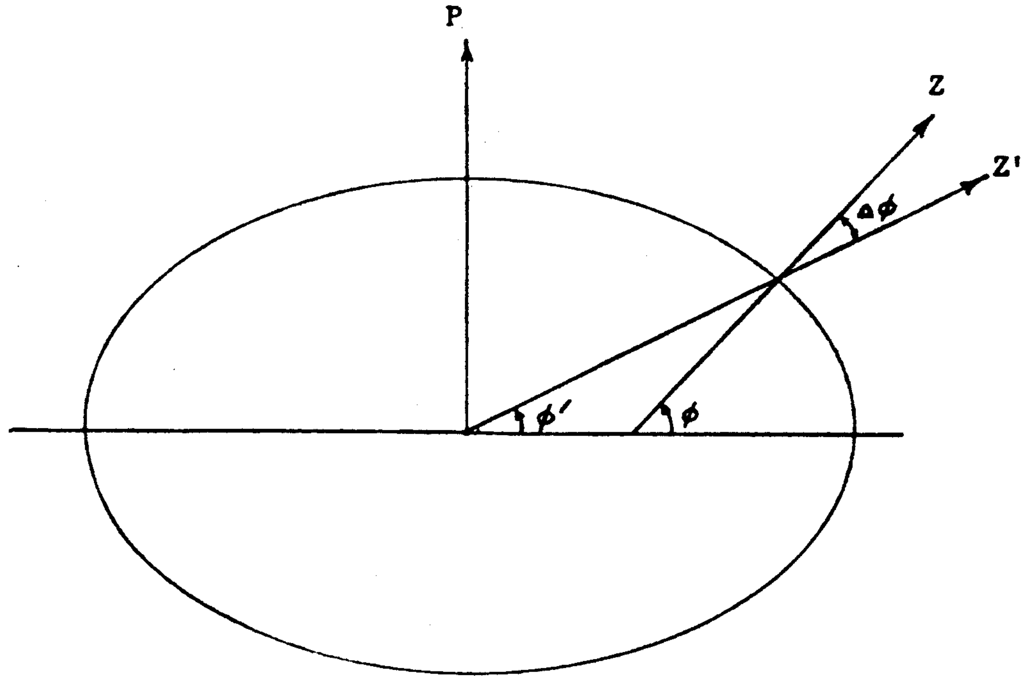


Figure 8

The angle $\Delta \phi$ between the geocentric and geodetic zeniths is simply the difference in the geocentric and geodetic latitudes

$$\Delta \phi = \phi - \phi' \tag{118}$$

This difference can be related to the eccentricity e of the elliptical cross section by

$$\tan \Delta \phi = \frac{e^2 \sin \phi \cos \phi}{1 - e^2 \sin^2 \phi} \tag{119}$$

or by

$$\tan \Delta \phi = \frac{e^2 \sin \phi' \cos \phi'}{1 - e^2 \cos^2 \phi'} \tag{120}$$

The projection of P, Z and Z' on the celestial sphere along with an object X is shown below. It is immediately apparent that neither the declination δ nor the hour angle H of X is affected by the displacement of the zenith through the angle $\Delta \phi$. However, corrections must be applied to both the computed elevation angle γ and azimuth angle σ to yield the geodetic oriented values γ (geod) and σ (geod) respectively.

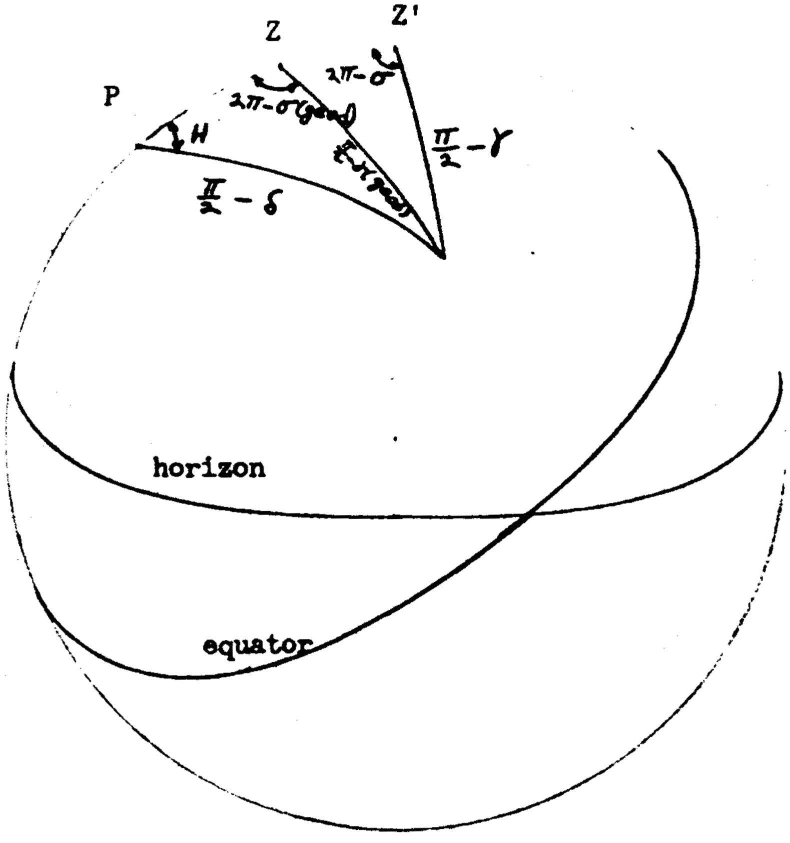


Figure 9

The geodetic azimuth angle can be found immediately by using the geodetic latitude in the transformations between (L_x, L_y, L_z) and (L_{hx}, L_{hy}, L_{hz}). Thus from equations 51, 52 and 53

$$\tan \sigma\ (\text{geod}) = \frac{L_x \sin \Theta - L_y \cos \Theta}{L_x \sin \phi \cos \Theta + L_y \sin \phi \sin \Theta - L_z \cos \phi} \tag{121}$$

In other words the second rotation described in section IIIB is through the angle ϕ instead of ϕ'.

Similarly the geodetic elevation angle γ (geod) is obtained from eq. 54 with ϕ substituted for ϕ'.

$$\sin \gamma\ (\text{geod}) = L_x \cos \phi \cos \Theta + L_y \cos \phi \sin \Theta + L_z \sin \phi \tag{122}$$

The corrections from the geodetic zenith z to the local zenith z_L are illustrated by Figure 10. Because the local zenith does not necessarily lie in the meridian plane $P\,Z\,Z'$, the deviations from Z are described by a north south component u, measured positively toward the north and an east west component v, measured positively toward the west. The latitude ϕ_a associated with the local zenith Z_L is called the astronomical latitude.

The correction $\Delta_v H = H_L - H$ to the hour angle H is obtained directly by applying the law of sines to the triangle P A Z_L.

$$\sin \Delta_v H = - \sin v \sec \phi_a \tag{123}$$

To the first order equation 123 reduces to

$$\Delta_v H = -v \sec \phi \tag{124}$$

There is no correction to the right ascension α or declination δ. From Figure 10 the first order correction to the latitude can be written

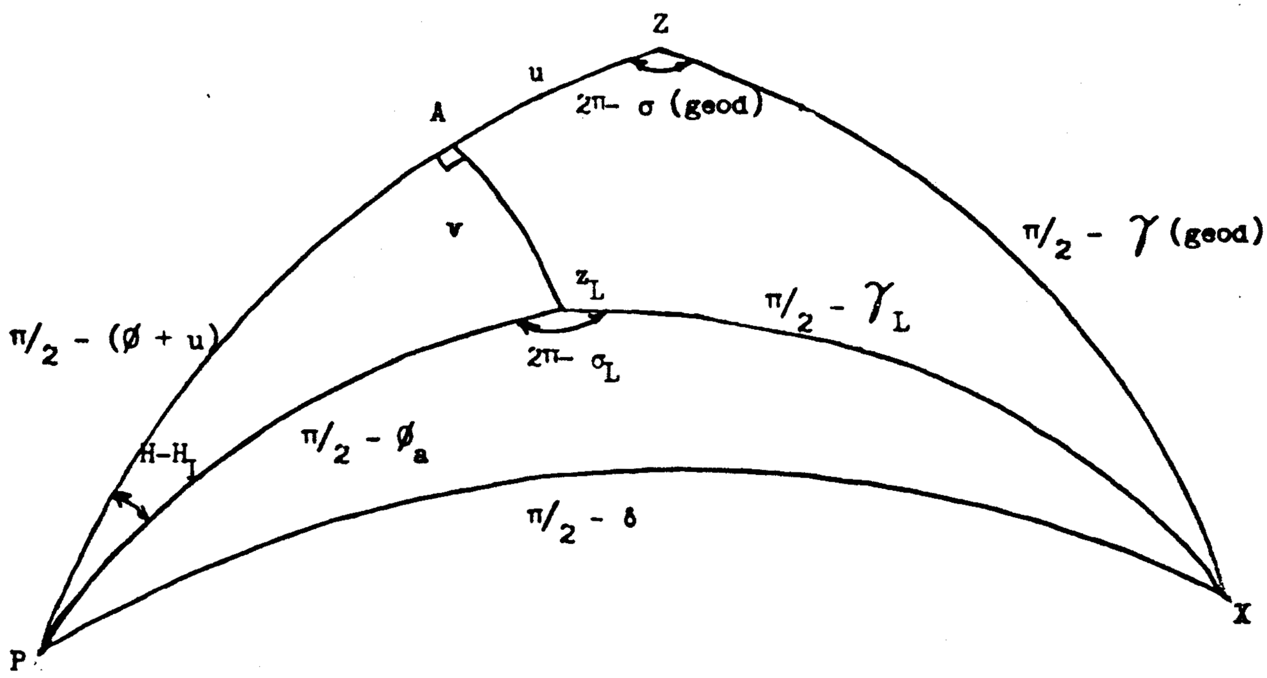


Figure 10

$$\Delta_v \phi = \phi_a - \phi = u \tag{125}$$

The correction to the elevation angle γ is obtained by differentiating L_{hz} in equations 51 and 54. Note that $\Delta_v L_x$, $\Delta_v L_y$ and $\Delta_v L_z$ are zero because $\Delta_v \alpha$ and $\Delta_v \delta$ are zero.

$$\cos \gamma \, d \gamma = d L_{hz} \tag{126}$$

and
$$d L_{hz} = - L_{hx} \, d \phi + L_{hy} \cos \phi \, d \Theta \tag{127}$$

Substitute equations 124 and 125 into the above and use eq. 51 to obtain

$$\Delta_v \gamma = u \cos \sigma - v \sin \sigma \tag{128}$$

Differentiate L_{hx} to obtain $\Delta_v \sigma$, the correction to the azimuth angle.

$$d L_{hx} = \sin \gamma \cos \sigma \, d \gamma + \cos \gamma \sin \sigma \, d \sigma \tag{129}$$

or

$$d L_{hx} = L_{hz} \, d \phi + L_{hy} \sin \phi \, d \theta \tag{130}$$

Therefore

$$\cos \gamma \, \Delta_v \sigma = u \sin \gamma \sin \sigma - v (\tan \phi \cos \gamma - \sin \gamma \cos \sigma) \tag{131}$$

C. Refraction

The correction to the observables caused by the bending of an electromagnetic wave in the atmosphere is perhaps the most unsatisfying of all the corrections, because the atmosphere is not static and fluctuations will cause unknown variable errors in any corrective formula. It is therefore necessary to resort to mean corrections based on some reasonable model of the atmosphere. Also refraction has an effect on all the observables and the only way to improve the accuracy of radio measurements beyond the limitations imposed by the atmosphere is to establish stations in space.

The first assumption in deriving a refraction correction is that the wave is confined to a plane containing the observer, the object and the center of the earth. In other words a signal is sent from P and arrives at S (Fig. 11).

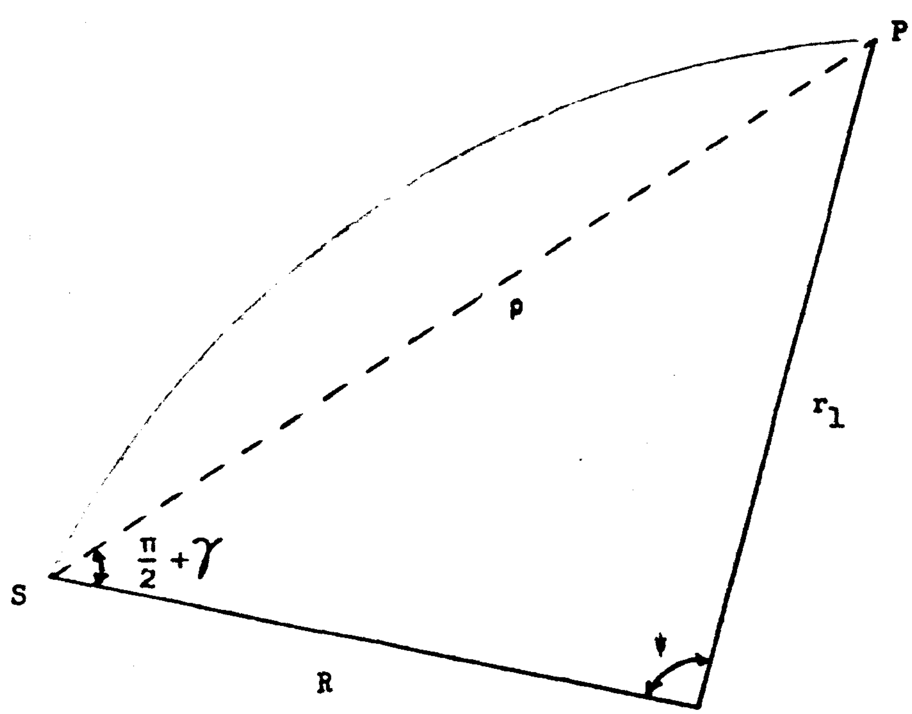


Figure 11

The time required for the signal to travel between these two points is designated by τ, and if the velocity of propagation is given by c, then clearly in the absence of an atmosphere

$$\tau = \frac{\rho}{c} \tag{132}$$

where ρ is the distance between S and P. However, if an atmosphere is introduced, the velocity of propagation will no longer be the constant c but will instead be a variable v. The ratio of c to v is called the index of refraction n, which for empty space is identically equal to unity.

$$n = \frac{c}{v} \tag{133}$$

For the case where n is a variable the time of transmission τ is given by

$$\tau = \int_S^P \frac{ds}{v} \tag{134}$$

The element of arc length ds is expressed in terms of the polar coordinates r and ψ of Figure 11 by

$$ds^2 = dr^2 + r^2 d\psi^2 \tag{135}$$

or with $p = \frac{d\psi}{dr}$ (136)

$$\frac{ds}{dr} = \sqrt{1 + r^2 p^2} \tag{137}$$

Therefore the time of transmission can be written as the integral

$$\tau = \frac{1}{c} \int_R^{r_1} n \sqrt{1 + r^2 p^2} \; dr \tag{138}$$

As a matter of interpretation the coordinate r_1 of Figure 11 is the geocentric distance to the transmitter at P. Thus the altitude H of P above a sphere passing through the receiver S is

$$H = r_1 - R \tag{139}$$

The index of refraction n in equation 138 is simply a function of the physics of the atmosphere and must be chosen once and for all from a consideration of atmospheric measurements. On the other hand the function p is arbitrary and for each function selected a different value of the time of transmission τ can result. Therefore, in order to specify p, a physical law is introduced known as Fermat's principal, which states that of all possible paths, a wave will follow the particular path that makes the time of transmission a minimum. From the calculus of variations the integral eq. 138 is a minimum if the following relation is satisfied.

$$\frac{\partial f}{\partial \Psi} - \frac{d}{dr} \frac{\partial f}{\partial p} = 0 \tag{140}$$

where the function f is the integrand of eq. 138

$$f = n \sqrt{1 + r^2 p^2} \tag{141}$$

The restriction that the index of refraction n is independent of the angle ψ is now applied so that the quantity $\partial f/\partial p$ is a constant k.

$$\frac{\partial f}{\partial p} = \frac{n\, r^2\, p}{\sqrt{1 + r^2 p^2}} = k \tag{142}$$

An evaluation of this constant at r = R yields

$$k = \frac{n_o R^2 p_o}{\sqrt{1 + R^2 p_o^2}} \tag{143}$$

To obtain p_o consider Fig. 11. In terms of the elevation angle γ the law of sines gives the relation

$$R \cos \gamma = r \cos (\gamma + \psi) \tag{144}$$

and differentiating with respect to the radius r yields

$$r\, p \sin (\gamma + \psi) = \left[R \sin \gamma - r \sin (\gamma + \psi)\right] \frac{d\gamma}{dr} + \cos (\gamma + \psi) \tag{145}$$

When r = R eq. 145 reduces to

$$R\, p_o = \text{ctn}\, \gamma\, (\text{ob}) \tag{146}$$

where γ (ob) is the value of γ when the wave reaches S, or in other words, γ(ob) is the observed elevation angle. Thus

$$k = n_o R \cos \gamma (\text{ob}) \tag{147}$$

and the function p that minimizes the time of transmission is

$$\frac{d\Psi}{dr} = \frac{n_o R \cos \gamma(ob)}{r\sqrt{n^2 r^2 - n_o^2 R^2 \cos^2 \gamma(ob)}} \tag{148}$$

With n given as a function of the radius r, eq. 148 can be integrated.

$$\Psi = n_o R \cos \gamma(ob) \int_R^{r_1} \frac{dr}{r\sqrt{n^2 r^2 - n_o^2 R^2 \cos^2 \gamma(ob)}} \tag{149}$$

Practically Ψ is obtained by a numerical evaluation of the indicated integral and then by Fig. 11 the uncorrected elevation angle γ is obtained.

$$\tan \gamma = \frac{r_1 \cos \Psi - R}{r_1 \sin \Psi} \tag{150}$$

The result is that a correction to the elevation angle is formed by differencing the observed and computed angles.

$$\Delta_r \gamma = \gamma(ob) - \gamma \tag{151}$$

Similarly a correction $\Delta_r \rho$ to the range can be obtained. A consideration of section IIIA indicates that the observed range ρ (ob) is simply

$$\rho\,(ob) = c\,\tau \tag{152}$$

Therefore, from eq. 138

$$\rho\ (\text{ob}) = \int_{R}^{r_1} n \sqrt{1 + r^2 p^2}\ dr \tag{153}$$

with p given by eq. 148. An attempt to evaluate the integral of eq. 153 and to form the difference of ρ (ob) and ρ leads to numerical difficulties in the subtraction of the two large quantities. However, it is a simple matter to derive the variation

$$\frac{d\,\Delta_r \rho}{dr} = \frac{d\,\rho\ (\text{ob})}{dr} - \frac{d\rho}{dr} \tag{154}$$

Using equations 17 and 22 the variation $d\rho$ (ob)/dr can be written

$$\frac{d\rho\ (\text{ob})}{dr} = \frac{n^2 r^2 p}{n_o R \cos \gamma (\text{ob})} \tag{155}$$

The variation $d\rho/dr$ is derived from the law of cosines applied to Fig. 11.

$$\rho^2 = r^2 + R^2 - 2rR \cos \psi \tag{156}$$

so that

$$\rho \frac{d\rho}{dr} = (r - R \cos \psi) + r R \sin \psi \frac{d\psi}{dr} \tag{157}$$

At ρ_o where r = R and ψ = 0, this expression degenerates, and therefore the elevation angle is used instead of the angle ψ in eq. 157. From Fig. 11.

$$r - R\cos\psi = \frac{\rho}{r}\sqrt{r^2 - R^2\cos^2\gamma} \tag{158}$$

$$r\sin\psi = \rho\cos\gamma \tag{159}$$

Thus the range ρ cancels throughout eq. 157 and

$$\frac{d\rho}{dr} = p\,R\cos\gamma + \sqrt{1 - \frac{R^2\cos^2\gamma}{r^2}} \tag{160}$$

The variation of eq. 154 follows immediately by subtracting eq. 160 from 155.

$$\frac{d\,\Delta_r\,\rho}{dr} = p\left(\frac{r^2 n^2}{R n_o \cos\gamma(ob)} - R\cos\gamma\right) - \sqrt{1 - \frac{R^2\cos^2\gamma}{r^2}} \tag{161}$$

The correction $\Delta_r\dot{\rho}$ to range rate is defined by $\Delta_r\dot{\rho} = \dot{\rho}\,(ob) - \dot{\rho}$ and so by the definition of $\Delta_r\,\rho$

$$\Delta_r\dot{\rho} = \frac{d}{dt}\,\Delta_r\,\rho \tag{162}$$

Now recognizing that the computed elevation angle γ is a function of n_o, n, R, γ(ob) and r through the integral of eq. 149, it follows by eq. 161 that $\Delta_r\,\rho$ is also a function of these parameters. Further, if there are no time varying quantities in n except the radius r, then the only time varying parameters in $\Delta_r\,\rho$ are the observed elevation angle γ(ob) and the radius. Therefore,

$$\Delta_r \dot{\rho} = \frac{\partial \Delta_r \rho}{\partial r} \frac{dr}{dt} + \frac{\partial \Delta_r \rho}{\partial \gamma(ob)} \frac{d\gamma(ob)}{dt} \tag{163}$$

The partial derivative of $\Delta_r \rho$ with respect to r in eq. 163 assumes that $\gamma(ob)$ is held constant. However, this is precisely the interpretation placed on eq. 161 and so this variation is already available. The second variation $\partial \Delta_r \rho/\partial \gamma(ob)$ is not available and although it could be obtained numerically from the preceding formulas, it is more usual to perform a numerical differentiation of $\Delta_r \rho$ with respect to $\gamma(ob)$ for various values of r.

As an example of the calculation of refraction corrections to elevation angle γ, range ρ and range rate $\dot{\rho}$, consider an exponential model for the index of refraction n.

$$n = 1 + (n_o - 1)\, e^{\left(\frac{-r-R}{S}\right)} \tag{164}$$

Typical values for constants $n_o - 1$ and the scale height S are

$$n_o - 1 = 3.40 \times 10^{-4}$$

$$S = 7.315 \text{ km}$$

A step by step numerical integration of eq. 148 with the above model for n and an elevation of γ by eq. 150 yields Fig. 12 for the correction to the elevation angle. The results of the integration are plotted as a function of the height H for several values of the observed elevation angle $\gamma(ob)$. Fig. 13 shows the values of the correction $\Delta_r \gamma$ for infinite height plotted against the observed angle $\gamma(ob)$. Infinite height is here defined as the point where the derivative $d\psi/dr$ is essentially zero. Fig. 14 shows the integration of eq. 161 for the range correction and Fig. 15

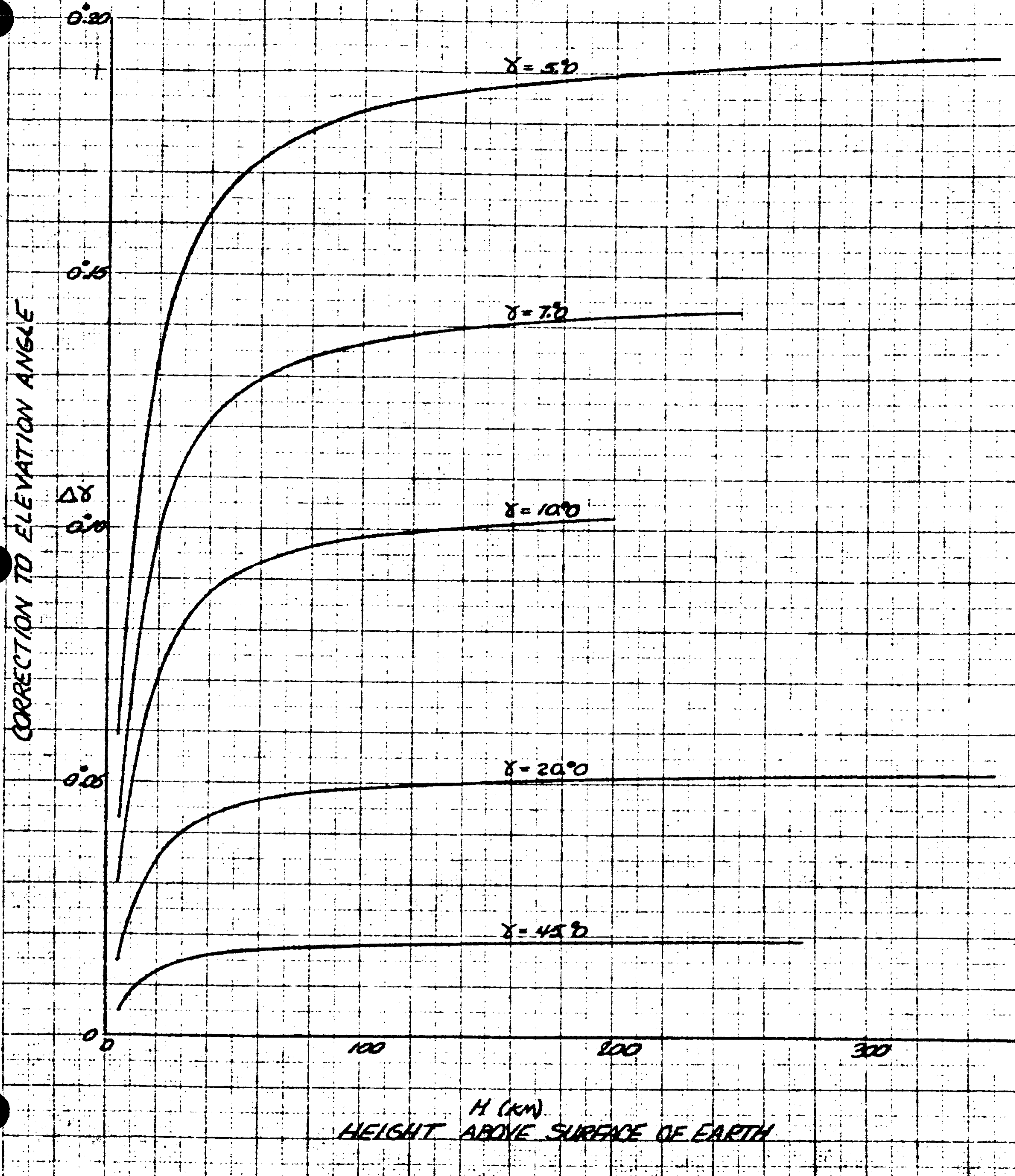

0.20
0.15
0.10
0.05
0
Δγ
CORRECTION TO ELEVATION ANGLE
γ = 5.0
γ = 7.0
γ = 10.0
γ = 20.0
γ = 45.0
0
100
200
300
H (KM)
HEIGHT ABOVE SURFACE OF EARTH

FIG. 12

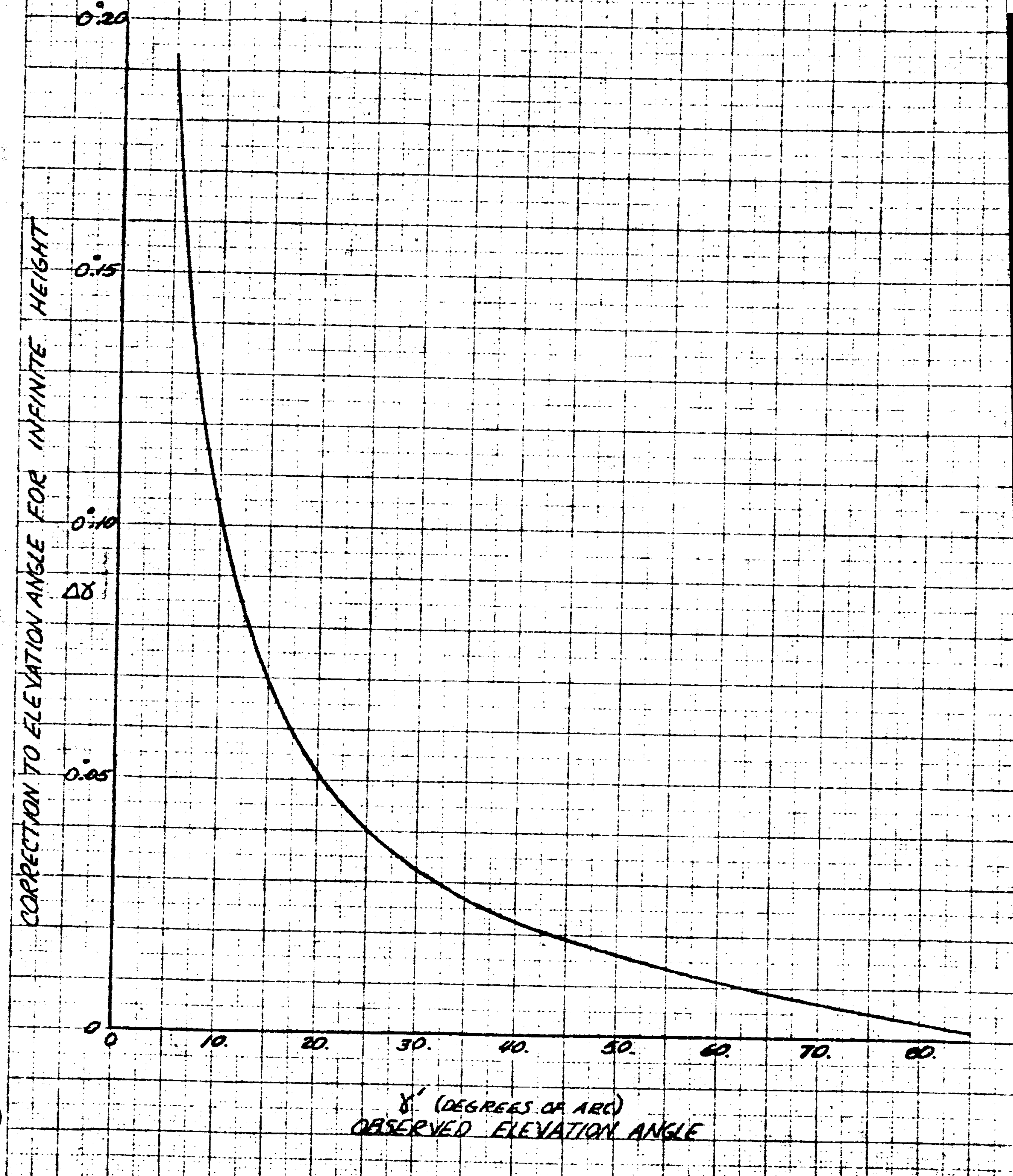

0.20°
0.15°
0.10°
0.05°
0
CORRECTION TO ELEVATION ANGLE FOR INFINITE HEIGHT
Δγ
0
10.
20.
30.
40.
50.
60.
70.
80.
γ' (DEGREES OF ARC)
OBSERVED ELEVATION ANGLE

FIG. 13

RE-ORDER No. 64-311

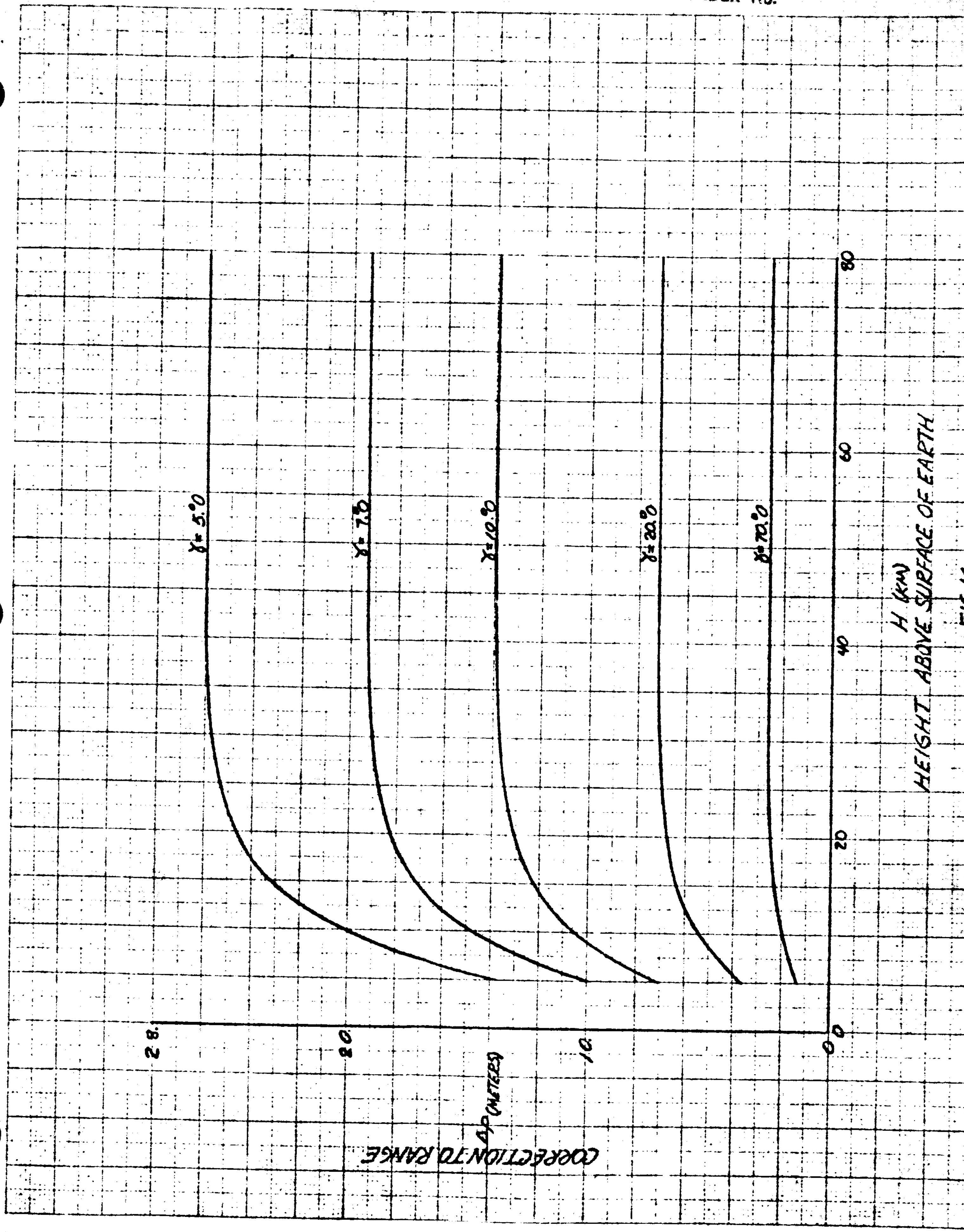


FIG 1A

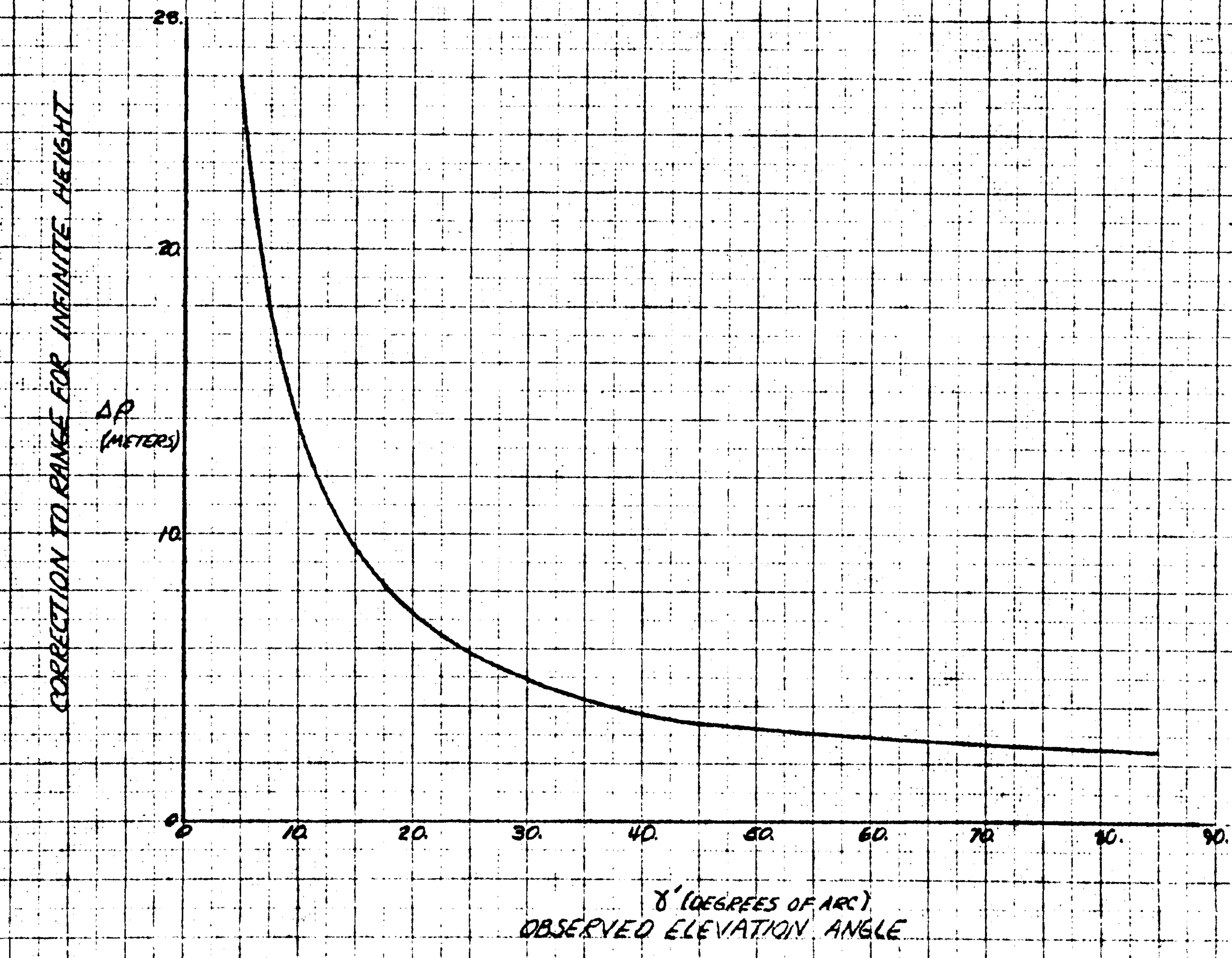
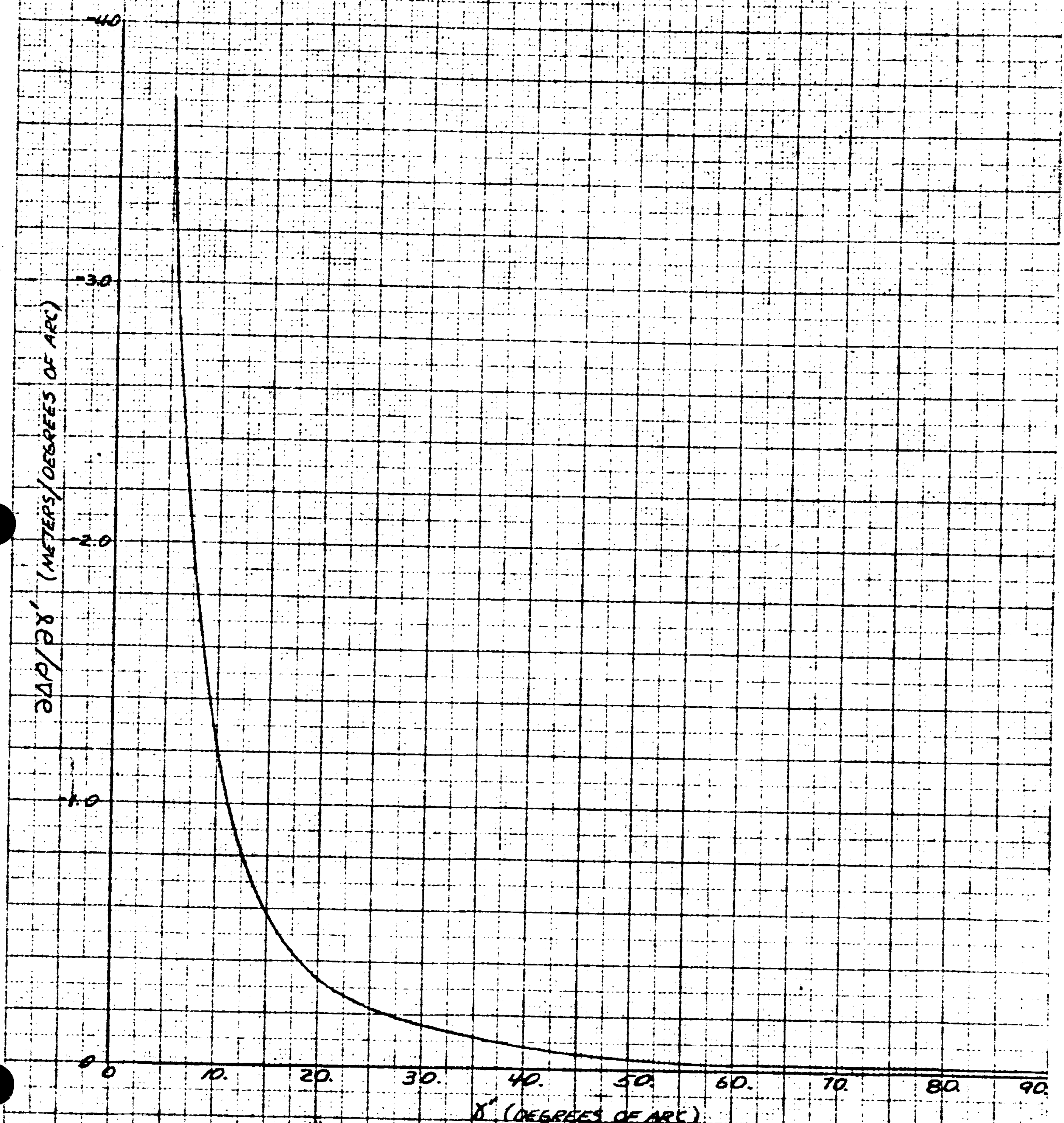

CORRECTION TO RANGE FOR INFINITE HEIGHT
ΔP
(METERS)
28.
20.
10.
0
0
10.
20.
30.
40.
50.
60.
70.
80.
90.
γ' (DEGREES OF ARC)
OBSERVED ELEVATION ANGLE

FIG. 15



OBSERVED ELEVATION ANGLE
FIG. 16

shows the corresponding correction at infinite height in a similar fashion to Figures 12 and 13 respectively. For the range rate correction the result is plotted for infinite height only. The variation $\partial \Delta_r \rho / \partial r$ is zero but a numerical differentiation of Fig. 15 yields $\partial \Delta_r \rho / \partial \gamma(ob)$ as a function of $\gamma(ob)$ according to Fig. 16. The corresponding correction to range rate can be obtained by multiplying $\partial \Delta_r \rho / \partial \gamma(ob)$ by the elevation angle rate $\dot{\gamma}(ob)$. For example, if $\dot{\gamma}(ob)$ is on the order of the angular rotation rate of the earth, the correction to range rate at an elevation angle of five degrees is about -0.015 meters per second. This is approximately the situation for range rate observations of a distant planet or space probe. For a near object such as an earth satellite, however, the correction to range rate can easily exceed one meter per second.

The final correction to the angular observations is obtained by a direct application of $\Delta_r \gamma$, the correction to the elevation angle. Because of the assumption that the signal from P to S travels only in the plane containing S, P and O, there is no correction to the azimuth angle σ. The correction to the declination angle δ is derived to the first order in $\Delta_r \gamma$ by differentiating L_z in eq. 46.

$$\cos \delta \, \Delta_r \delta = \Delta L_z \tag{165}$$

From equations 52, 53 and 54

$$L_z = - L_{hx} \cos \phi' + L_{hz} \sin \phi' \tag{166}$$

and $\Delta_r L_z = - \Delta_r L_{hx} \cos \phi' + \Delta_r L_{hz} \sin \phi'$ (167)

But from eq. 51

$$\Delta_r L_{hx} = \sin \gamma \cos \sigma \Delta_r \gamma \qquad (168)$$

$$\Delta_r L_{hz} = \cos \gamma \Delta_r \gamma \qquad (169)$$

and

$$\cos \delta \Delta_r \delta = (\cos \gamma \sin \phi' - \sin \gamma \cos \sigma \cos \phi') \Delta_r \gamma \qquad (170)$$

Similarly the correction to the hour angle is

$$\Delta_r H = - \Delta_r \alpha \qquad (171)$$

where

$$\Delta_r \alpha = \frac{\cos \phi \sin^2 H}{\cos^2 \gamma \sin \sigma} \Delta_r \gamma \qquad (172)$$

It should be pointed out that the above corrections are applied to direct angular measurements only. Thus for optical observations the refraction correction is applied to angles obtained through the use of an instrument such as a transit circle. For the more usual photographic measurements the refraction correction is applied differentially in the process of reducing the photographic plates to usable angular observations. Of course for radio observations the refraction correction is always applied.

For purposes of calculation in the ODP, empirical interpolation formulas for $\Delta_r \rho$, $\Delta_r \dot{\rho}$ and $\Delta_r \gamma$ are used in place of the actual curves generated in this section. They are given in T. M. 33-168.

D. Aberration

In addition to the light time correction described in section IVA, the apparent position of an object is further affected by the motion of the observer in space. Text books in spherical astronomy give the appropriate aberrational corrections to the right ascension α and declination δ for an earth based observer.

At present the accuracies in radio angles are not sufficient to include the aberrational correction and it is not included in the ODP. However, future observational refinements, especially in optical angular measurements from a manned vehicle, could make such a systematic correction sensible.

It should be noted that the aberrational correction to doppler frequencies was included in section IIID where the approach was to include relativistic effects in the derivation. In other words the aberrational effect is a direct consequence of the special theory of relativity

V. Regression Coefficients

In the application of the estimation formula eq. 11 the coefficients that relate variations in the parameters $\underline{x}$ to variations in the data $\underline{z}$ are required. Also in the formation of the parameter covariance matrix Γ_x, both the coefficients for the parameter set $\underline{x}$ and for the parameters $\underline{y}$, which are not estimated, are required. In other words the matrices A_x and A_y must be evaluated so that the nature of the variational expression eq. 173 is known.

$$\delta \underline{z} = A_x \, \delta \underline{x} + A_y \, \delta \underline{y} \tag{173}$$

For purposes of computing the variational coefficients it is not necessary to distinguish between the parameter sets $\underline{x}$ and $\underline{y}$ and to simplify the discussion we will consider the evaluation of the regression coefficient matrix A defined in terms of the totality of parameters $\underline{q}$.

$$\delta \underline{z} = A \, \delta \, \underline{q} \tag{174}$$

However, it is necessary to partition the parameters $\underline{q}$ into the following sets.

1. $\underline{X}$ State variables
2. $\underline{X}_o$ State variables at epoch t_o
3. $\underline{a}$ Total of all constants affecting the state $\underline{X}$
4. $\underline{b}$ Constants used explicitly in the computation of the data

In the ODP the state variables are always the true geocentric equatorial cartesian components of position and velocity. The parameters $\underline{a}$ are those constants that enter into the computation of the trajectory of the probe. For example the astronomical unit, the masses of various bodies in the solar system and a solar radiation pressure constant are included in $\underline{a}$. On the other hand constants such as the geocentric coordinates of the tracking stations are included in $\underline{b}$ and affect only the computation of the data themselves. Thus the collection of data represented by $\underline{z}$ can be written as a function of the state variables $\underline{X}$, constants $\underline{b}$ and the constants $\underline{a}$ which can enter into the calculation of $\underline{z}$ both explicitly and implicitly through the state variables.

$$\underline{z} = \underline{z}\,(\underline{X}, \underline{a}, \underline{b}) \tag{175}$$

In practice the data admissable to the ODP do not contain the constants $\underline{a}$ explicitly and eq. 175 can be written in the specialized form

$$\underline{z} = \underline{z}\,(\underline{X}, \underline{b}) \tag{176}$$

The variation in $\underline{z}$ is

$$\delta\underline{z} = A_X\,\delta\underline{X} + A_b\,\delta\underline{b} \tag{177}$$

However, the variation $\delta\underline{X}$ in the state variables is further related to variations in the initial conditions $\underline{X}_o$ and the constants $\underline{a}$ through the solution $\underline{X}$ to the equations of motion.

$$\underline{X} = \underline{X}\,(\underline{X}_o, \underline{a}) \tag{178}$$

The variations are given in terms of the state transition matrix U and the parameter sensitivity matrix V.

$$\delta\underline{X} = U\,\delta\,\underline{X}_o + V\,\delta\,\underline{a} \tag{179}$$

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When eq. 179 is substituted in eq. 177 the required data variations are given in terms of the parameter set $\underline{q}$ and the regression coefficient matrix A is specified.

$$\delta \underline{z} = A_X U \delta \underline{X}_o + A_X V \delta \underline{a} + A_b \delta \underline{b} \tag{180}$$

Clearly the matrices A_X and A_b can be evaluated through formulas obtained by forming the differentials of the various expressions for the observables as given in section III. These differentials are given in section VB. However because the solution $\underline{X}$ to the equations of motion can not be written in closed form it is necessary either to find approximate expressions for the U and V matrices, evaluate them by finite differencing or evaluate them along with the equations of motion by numerical integration. It is the latter method that is used by the ODP. In other words the equations of motion can be written in the form

$$\dot{\underline{X}} = \dot{\underline{X}} (\underline{X}, \underline{a}) \tag{181}$$

and the first variation in $\dot{\underline{X}}$ is

$$\delta \dot{\underline{X}} = \Theta \delta \underline{X} + \Phi \delta \underline{a} \tag{182}$$

where expressions for Θ and Φ can be developed easily once the equations of motion are specified. They are given in section VA. Substitute eq. 179 into eq. 182 to obtain the variation in the equations of motion in terms of variations in the parameter set $\underline{q}$.

$$\delta \dot{\underline{X}} = \Theta U \delta \underline{X}_o + (\Theta V + \Phi) \delta \underline{a} \tag{183}$$

Also eq. 179 can be differentiated with respect to time in order to obtain the same variation.

$$\delta \underline{\dot{X}} = \dot{U}\, \delta\, \underline{X}_o + \dot{V}\, \delta\, \underline{a} \tag{184}$$

Thus in order that the variations in the parameters be arbitrary, it is necessary that the coefficients in equations 183 and 184 be equal.

$$\dot{U} = \Theta U \tag{185}$$

$$\dot{V} = \Theta V + \Phi \tag{186}$$

In the ODP equation 185 is numerically integrated step by step along with the equations of motion, eq. 181. Note that the solution to eq. 185 is dependent on the solution to eq. 181 because the matrix Θ is a function of the state variables $\underline{X}$. The solution to eq. 186 could also be obtained along with $\underline{X}$ and U by the step by step numerical integration. However in the present version of the ODP the matrix V is evaluated by quadrature formulas in a separate computation. Thus the solution to eq. 186 is written in the form

$$V(t) = U(t) \int_o^t U^{-1}(\tau)\, \Phi(\tau)\, d\tau \tag{187}$$

To verify that this is a solution to eq. 186 differentiate V(t) with respect to time

$$\dot{V} = \dot{U} \int_o^t U^{-1}\, \Phi\, d\tau + U\, U^{-1}\, \Phi \tag{188}$$

But from eq. 187 the integral $\int_0^t U^{-1} \phi \, d\tau$ is given by

$$\int_0^t U^{-1} \phi \, d\tau = U^{-1} V \tag{189}$$

and with equations 185 and 189 substituted into eq. 188 it simplifies to

$$\dot{V} = \Theta V + \phi \tag{190}$$

which is equivalent to eq. 186

To summarize, the regression coefficient matrix A is computed by the following procedure.

1. From the numerical integration of equations 181, 185 and 187 obtain the matrices U and V at the times associated with the data set z. The expressions of section VB are used for the evaluation of Θ and ϕ required for the integration.
2. From the formulas of section VA evaluate the matrices A_X and A_b at the data times.
3. Form the matrix products A_X U and A_X V and collect the results to obtain the matrix A as indicated by eq. 180

Of course in practice the matrices A_X U, A_X V, A_b and finally A are constructed one row at a time where each row corresponds to a particular type of observation at a particular time of observation.

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A. Variations in the Accelerations

In this section expressions are derived for the elements of the matrices Θ and Φ which are simply the linear coefficients in the variational relationship of eq. 182. It is convenient in specifying the elements to partition the vector $\underline{X}$ into the position coordinates $\underline{r}$ and the velocity coordinates $\dot{\underline{r}}$. Then eq. 182 can be written as follows

$$\begin{pmatrix} \delta\,\dot{\underline{r}} \\ \cdots\cdots \\ \delta\,\ddot{\underline{r}} \end{pmatrix} = \Theta \begin{pmatrix} \delta\,\underline{r} \\ \cdots\cdots \\ \delta\,\dot{\underline{r}} \end{pmatrix} + \Phi\,\delta\,\underline{a} \tag{191}$$

It is also convenient to introduce the following notation. Suppose a matrix contains as elements a set of partial derivatives so that the element in the i^{th} row and j^{th} column of the matrix is $\partial x_i/\partial y_j$. Then we designate the matrix itself by the notation $(\partial\underline{x}/\partial\underline{y})$. Thus the partitioned form of the Θ and Φ matrices appears as given below.

$$\Theta = \begin{pmatrix} \left(\dfrac{\partial\,\dot{\underline{r}}}{\partial\,\underline{r}}\right) & \vdots & \left(\dfrac{\partial\,\dot{\underline{r}}}{\partial\,\dot{\underline{r}}}\right) \\ \cdots\cdots & \cdot & \cdots\cdots \\ \left(\dfrac{\partial\,\ddot{\underline{r}}}{\partial\,\underline{r}}\right) & \vdots & \left(\dfrac{\partial\,\ddot{\underline{r}}}{\partial\,\dot{\underline{r}}}\right) \end{pmatrix} \tag{192}$$

$$\Phi = \begin{pmatrix} \left(\dfrac{\partial\,\dot{\underline{r}}}{\partial\,\underline{a}}\right) \\ \cdots\cdots \\ \left(\dfrac{\partial\,\ddot{\underline{r}}}{\partial\,\underline{a}}\right) \end{pmatrix} \tag{193}$$

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The six matrices in equations 192 and 193 can be evaluated by writing the equations of motion, eq. 181, in the expanded form. First designate the components of $\underline{X}$ by $(X_1, X_2, X_3, X_4, X_5, X_6)$. Then

$$\begin{aligned} \dot{X}_1 &= \dot{x} \\ \dot{X}_2 &= \dot{y} \\ \dot{X}_3 &= \dot{z} \end{aligned} \tag{194}$$

$$\begin{aligned} \dot{X}_4 &= \ddot{x}\ (\underline{r}, \underline{a}) \\ \dot{X}_5 &= \ddot{y}\ (\underline{r}, \underline{a}) \\ \dot{X}_6 &= \ddot{z}\ (\underline{r}, \underline{a}) \end{aligned} \tag{195}$$

and it is obvious that if the matrices in equations 192 and 193 represent the coefficients in eq. 182, then the matrices $(\partial \dot{\underline{r}}/\partial \underline{r})$ and $(\partial \dot{\underline{r}}/\partial \underline{a})$ are always null and for conservative force fields the matrix $(\partial \ddot{\underline{r}}/\partial \dot{\underline{r}})$ is null also. In addition, the matrix $(\partial \dot{\underline{r}}/\partial \dot{\underline{r}})$ is the 3 x 3 unit matrix and therefore the only matrices which are non trivial to evaluate are $(\partial \ddot{\underline{r}}/\partial \underline{r})$ and $(\partial \ddot{\underline{r}}/\partial a)$. Taken together these two matrices describe the total first variation in the accelerations of the probe.

For the equations of motion used in the ODP the accelerations can be written in a fairly compact form by employing a vector $\underline{h}(\underline{x})$ defined by equation 196.

$$\underline{h}\ (\underline{x}) = - \frac{\underline{x}}{x^3} \tag{196}$$

where x is the euclidean norm or magnitude of the vector $\underline{x}$

$$x^2 = \underline{x} \cdot \underline{x} \tag{197}$$

The geocentric accelerations are then written in the vector form

$$\ddot{\underline{r}} = \mu_E \underline{h}(\underline{r}) + \mu_M \left[\underline{h}(\underline{r} - \underline{r}_☾) + \underline{h}(\underline{r}_☾) \right]$$

$$+ \mu_S \left[\underline{h}(\underline{r} - \underline{r}_\odot) + \underline{h}(\underline{r}_\odot) \right]$$

$$+ \sum_p \mu_p \left[\underline{h}(\underline{r} - \underline{r}_p) + \underline{h}(\underline{r}_p) \right]$$

$$+ \sum_{n=2}^{\infty} J_n \underline{g}_n (\underline{r}, \mu_E, a_E) - \frac{C_1 A_{rad}}{m} (1 + \gamma B) \underline{h}(\underline{r} - \underline{r}_\odot) \qquad (198)$$

In the ODP the equations of motion are always referred to the geocenter and thus the first term in eq. 198 is the two body acceleration on the probe by the earth. The second and third terms represent respectively the lunar and solar perturbations while the fourth gives the perturbative accelerations of the planets. The fifth term represents the bulge perturbations which result from the failure of the earth to act like a mass point and the last term is the perturbative acceleration from solar radiation pressure. The Poynting Robertson force is neglected. The constants in eq. 198 are the various gravitational constants for each body concerned and are defined as the product of the universal gravitational constant and the mass of the body. Also the zonal harmonic coefficients are given by $(J_1, J_2, \ldots)$ and are defined in terms of the assumed potential function for the earth.

$$U_E = \frac{\mu_E}{r} \left\{ 1 - \sum_{n=2}^{\infty} J_n \left(\frac{a_E}{r} \right)^n P_n (\sin \delta) \right\} \qquad (199)$$

where $\sin \delta = z/r$, a_E is the mean equatorial radius of the earth and $P_n(x)$ is the

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n^{th} order legendre polynomial. The constants in the radiation perturbation can be taken together as a single proportionality constant. However, in forming the variations the quantity γB will be allowed to vary rather than the whole constant. This is done for convenience since γB is dimensionless and is restricted to the interval $0 \leq \gamma B \leq 1$. It is thus easily interpreted when corrected by the estimation procedure. The constant γ is the albedo of the effective area A_{rad} of the spacecraft and B is a constant that depends on the reflection law of the effective area. The mass of the spacecraft is m and c_1 is a constant related to the solar constant.

In addition to the equations of motion being restricted by the ODP to the geocentric form, a further restriction is imposed in that the equations are in units of kilometers and seconds of time. However, the solar and planetary ephemerides are given in units of astronomical units while the lunar ephemeris is given in units of earth radii. Therefore, two more constants are imbedded in eq. 198, the conversion factor A_E which is the number of kilometers in an astronomical unit and R_E which is the number of kilometers in an earth radius. The ephemerides available to the ODP are assumed perfectly accurate and the only way to change the positions of the moon, sun and planets is by changing A_E and R_E. This situation is unsatisfactory for certain problems, especially where information on astronomical constants is being extracted from probe tracking data. However, in the version of the ODP described in T. M. 33-168 this is the situation and the derivations presented here are based on the assumption that the ephemerides are perfectly accurate. As a consequence the earth radius a_E used in the potential function for the earth is not necessarily the same number as the earth radius R_E used to scale the lunar ephemeris. In the program a_E is a fixed constant and any variation in the harmonics of the earth's potential function is absorbed entirely in the coefficients J_2, J_3 and J_4.

Finally to completely specify the equations of motion the form of the vectors g_n are required. By taking the gradient of eq. 199 they can be identified easily.

$$g_{x2} = \frac{3}{2} \frac{\mu_E x}{r^3} \left(\frac{a_E}{r} \right)^2 (1 - 5 \sin^2 \delta) \qquad x \to y \tag{200}$$

$$g_{z2} = \frac{3}{2} \frac{\mu_E z}{r^3} \left(\frac{a_e}{r} \right)^2 (3 - 5 \sin^2 \delta) \tag{201}$$

$$g_{x3} = \frac{5}{2} \frac{\mu_E x}{r^3} \left(\frac{a_E}{r} \right)^3 (3 - 7 \sin^2 \delta) \sin \delta \qquad x \to y \tag{202}$$

$$g_{z3} = -\frac{3}{2} \frac{\mu_E}{r^2} \left(\frac{a_E}{r} \right)^3 (1 - 10 \sin^2 \delta + \frac{35}{3} \sin^4 \delta) \tag{203}$$

$$g_{x4} = -\frac{5}{8} \frac{\mu_E x}{r^3} \left(\frac{a_E}{r} \right)^4 (3 - 42 \sin^2 \delta + 63 \sin^4 \delta) \qquad x \to y \tag{204}$$

$$g_{z4} = -\frac{5}{8} \frac{\mu_E z}{r^3} \left(\frac{a_E}{r} \right)^4 (15 - 70 \sin^2 \delta + 63 \sin^4 \delta) \tag{205}$$

Before forming the variation in eq. 198 the variation in the vector $\underline{h}$ $(\underline{x})$ of eq. 196 is derived so that as a result the required variation $\delta \ddot{\underline{r}}$ can be more easily obtained. Define the matrix $H(\underline{x})$ by

$$\delta \, \underline{h} \, (\underline{x}) = H \, (\underline{x}) \, \delta \, \underline{x} \tag{206}$$

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Then the elements of the matrix H in terms of $\underline{x} = (x_1, x_2, x_3)$ are

$$H_{ij}(\underline{x}) = \frac{\partial h_i(\underline{x})}{\partial x_j} = \frac{3\, x_i\, x_j}{x^5} - \frac{\delta_{ij}}{x^3} \tag{207}$$

where δ_{ij} is the Kronecker delta symbol.

Thus the variation of eq. 198 is given by

$$\delta\ddot{\underline{r}} = \mu_E\, H(\underline{r})\,\delta\,\underline{r} + \mu_M\left[H(\underline{r} - \underline{r}_{☾})(\delta\underline{r} - \delta\underline{r}_{☾}) + H(\underline{r}_{☾})\,\delta\underline{r}_{☾}\right]$$

$$+ \mu_s\left[H(\underline{r} - \underline{r}_{\odot})(\delta\underline{r} - \delta\underline{r}_{\odot}) + H(\underline{r}_{\odot})\,\delta\underline{r}_{\odot}\right]$$

$$+ \sum_p \mu_p\left[H(\underline{r} - \underline{r}_p)(\delta\underline{r} - \delta\underline{r}_p) + H(\underline{r}_p)\,\delta\underline{r}_p\right]$$

$$+ \sum_{n=2}^{4}\left[J_n\,\delta\,\underline{g}_n + \underline{g}_n\,\delta\,J_n\right]$$

$$- \frac{C_1\, A_{rad}}{m}(1 + \gamma B)\, H(\underline{r} - \underline{r}_{\odot})(\delta\underline{r} - \delta\underline{r}_{\odot})$$

$$- \frac{C_1\, A_{rad}}{m}\,\underline{h}(\underline{r} - \underline{r}_{\odot})\,\delta(\gamma B)$$

$$+ \underline{h}(\underline{r})\,\delta\,\mu_E + \left[\underline{h}(\underline{r} - \underline{r}_{☾}) + \underline{h}(\underline{r}_{☾})\right]\delta\,\mu_M \qquad +$$

$$+\left[\underline{h}\,(\underline{r}-\underline{r}_\odot)+\underline{h}\,(\underline{r}_\odot)\right]\delta\mu_s$$

$$+\sum_p\left[\underline{h}\,(\underline{r}-\underline{r}_p)+\underline{h}\,(\underline{r}_p)\right]\delta\mu_p \tag{208}$$

The variations in the vectors $\underline{g}_n$ are neglected for $n > 2$ and the required variations in $\underline{g}_2$ are obtained from equations 200 and 201. An additional matrix $G(\underline{r})$ is introduced such that

$$\delta\,\underline{g}_2 = G\,(\underline{r})\,\delta\underline{r} + \underline{g}_2\,\frac{\delta\,\mu_E}{\mu_E} \tag{209}$$

The elements of the matrix $G(\underline{r})$ are given in TR 32-223, p 81. Note that $G(\underline{r})$ is a symmetrical matrix.

Also in forming the Θ and ϕ matrices the solar radiation term is assumed zero in the $(\delta\underline{r} - \delta\underline{r}_\odot)$ variation. Therefore, the matrix $(\partial\ddot{\underline{r}}/\partial\underline{r})$ from an inspection of eq. 208 is given by

$$\left(\frac{\partial\ddot{\underline{r}}}{\partial\underline{r}}\right) = \mu_E\,H\,(\underline{r}) + J_2\,G\,(\underline{r}) + \mu_M\,H\,(\underline{r}-\underline{r}_{\text{☾}})$$

$$+\,\mu_s\,H\,(\underline{r}-\underline{r}_\odot) + \sum_p \mu_p\,H\,(\underline{r}-\underline{r}_p) \tag{210}$$

It is necessary to express the variations $\delta\underline{r}$, $\delta\underline{r}_o$ and $\delta\underline{r}_p$ in terms of constants that are available to the ODP before the matrix $(\partial\ddot{\underline{r}}/\partial\underline{a})$ can be written down. First of all consider the scaling of the lunar ephemeris.

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$$\underline{r}_{☾} = R_E \, \underline{r}_{☾} \text{ (radii)} \tag{211}$$

Then under the previous assumption that $\delta \, \underline{r}_{☾}$ (radii) = 0 the variation in $\underline{r}_{☾}$ is

$$\delta \, \underline{r}_{☾} = \underline{r}_{☾} \frac{\delta R_E}{R_E} \tag{212}$$

Similarly

$$\delta \, \underline{r}_{\odot} = \underline{r}_{\odot} \frac{\delta A_E}{A_E} \tag{213}$$

and

$$\delta \, \underline{r}_p = \underline{r}_p \frac{\delta A_E}{A_E} \tag{214}$$

Also the gravitational constants μ_s and μ_p are expressed in terms of A_E. By definition

$$\mu_s = k^2 A_E^3 \tag{215}$$

where k^2 is the Gaussian gravitational constant. Therefore,

$$\delta \, \mu_s = \frac{3 \, \mu_s}{A_E} \, \delta A_E \tag{216}$$

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For the planets the mass m_p of a planet in solar units is introduced. Then

$$\mu_p = m_p \mu_s \tag{217}$$

and

$$\delta \mu_p = \mu_s \delta m_p + 3 \mu_p \frac{\delta A_E}{A_E} \tag{218}$$

Now eq. 208 can be written in the form

$$\delta \ddot{\underline{r}} = \left(\frac{\partial \ddot{\underline{r}}}{\partial \underline{r}} \right) \delta \underline{r} + \frac{\mu_M}{R_E} \left[H(\underline{r}_☾) - H(\underline{r} - \underline{r}_☾) \right] \underline{r}_☾ \delta R_E$$

$$+ \sum_{i=1}^{n} \frac{\mu_i}{A_E} \left[H(\underline{r}_i) - H(\underline{r} - \underline{r}_i) \right] \underline{r}_i \delta A_E$$

$$+ \frac{J_2}{\mu_E} \underline{g}_2 \delta \mu_E + \sum_{n=2}^{4} \underline{g}_n \delta J_n$$

$$- \frac{C_1 A_{rad}}{m} \underline{h}(\underline{r} - \underline{r}_e) \delta (\gamma B) + \underline{h}(\underline{r}) \delta \mu_E$$

$$+ \left[\underline{h}(\underline{r} - \underline{r}_☾) + \underline{h}(\underline{r}_☾) \right] \delta \mu_M \qquad +$$

$$+ \sum_{i=1}^{n} \frac{3\,\mu_i}{A_E} \left[\underline{h}\,(\underline{r} - \underline{r}_i) + \underline{h}\,(\underline{r}_i) \right] \delta A_E$$

$$+ \mu_s \sum_p \left[\underline{h}\,(\underline{r} - \underline{r}_p) + \underline{h}\,(\underline{r}_p) \right] \delta m_p \tag{219}$$

where the summation on i occurs for the sun and the planets. Eq. 219 completely specifies the elements of the matrix $(\partial \bar{\underline{r}}/\partial \underline{a})$ as they are given in T. M. 33-168. Two subtleties of the above derivation should be noted.

First the solar ephemeris is given not in geocentric coordinates but instead with respect to the earth-moon barycenter. Thus if either μ_M or μ_E are changed the geocentric position of the barycenter will also change as will the geocentric ephemeris of the sun. Further because the planetary ephemerides are referred to the sun, a change in the geocentric ephemeris of the sun will be reproduced in the geocentric ephemerides of the planets. In other words equations 213 and 214 should rigorously include variations in μ_M and μ_E as well as the variation in A_E. However, these variations are neglected in accordance with the assumption that the geocentric ephemerides are perfectly accurate. In a more advanced program it would make sense to allow the variation in the ephemerides with μ_M and μ_E along with the sensitivity of the ephemerides to orbital elements of the earth, moon and planets.

The second subtlety arises from the condition that μ_E, μ_M and R_E cannot be changed without bounds because if μ_E and μ_M are given it is possible to compute the mean distance to the moon from dynamical considerations and a knowledge of the period of the moon. However, if R_E is given it is also possible to compute the mean distance simply by the scaling of the lunar ephemeris. This constraint between R_E, μ_E and μ_M could have been incorporated into the derivation of eq. 219 as was the similar constraint of eq. 215. However, the constraint of eq. 215 must be satisfied exactly as it in effect defines the astronomical unit. In contrast there is some uncertainty in the relation between R_E, μ_E and μ_M and a certain flexibility is retained by treating the variations in all three quantities. If one wishes to apply the constraint (eq. 220) it is done in the ODP during the formation of the regression matrix A.

$$R_E \propto (\mu_E + \mu_M)^{1/3} \tag{220}$$

Without the application of the constraint, R_E is considered as an independent variable so that the formula for the computation of a data type z is written

$$z = z\,(\mu_E,\ \mu_M,\ R_E) \tag{221}$$

Therefore, there exists three partial derivatives, a, b, and c.

$$a = \frac{\partial z}{\partial\,\mu_E} \tag{222}$$

$$b = \frac{\partial z}{\partial\,\mu_M} \tag{223}$$

$$c = \frac{\partial z}{\partial\,R_E} \tag{224}$$

Now when R_E is considered as a dependent variable the formula for z becomes a composite function of the form

$$z = z\left[\mu_E, \mu_M, R_E(\mu_E, \mu_M)\right] \tag{225}$$

and there exist two partial derivatives a' and b' which are obtained by using the rules for differentiating composite functions.

$$a' = a + c \frac{\partial R_E}{\partial \mu_E} \tag{226}$$

$$b' = b + c \frac{\partial R_E}{\partial \mu_M} \tag{227}$$

where from eq. 220

$$\frac{\partial R_E}{\partial \mu_E} = \frac{\partial R_E}{\partial \mu_M} = \frac{1}{3} \frac{R_E}{\mu_E + \mu_M} \tag{228}$$

B. Variations in the Observables

In order to obtain the matrices A_x and A_b defined by Eq. 177 it is useful to return to the definition of the range and range rate vectors of section III and form the variations of Eqs. 37 and 38.

$$\Delta \underline{\rho} = \Delta \underline{r} - \Delta \underline{R} \tag{229}$$

$$\Delta \underline{\dot{\rho}} = \Delta \underline{\dot{r}} - \Delta \underline{\dot{R}} \tag{230}$$

Thus if the variations in the observables can be expressed in terms of $\Delta \underline{\rho}$ and $\Delta \underline{\dot{\rho}}$, then Eqs. 229 and 230 provide the required elements for the A_x matrix as coefficients of $\Delta \underline{r}$ and $\Delta \underline{\dot{r}}$, and for the station coordinate elements of A_b through $\Delta \underline{R}$.

The variation in range is obtained immediately from Eq. 39

$$\rho \Delta \rho = \underline{\rho} \cdot \Delta \underline{\rho} \tag{231}$$

Similarly Eq. 40 can be differentiated to obtain $\Delta \dot{\rho}$.

$$\rho \Delta \dot{\rho} + \dot{\rho} \Delta \rho = \underline{\rho} \cdot \Delta \underline{\dot{\rho}} + \underline{\dot{\rho}} \cdot \Delta \underline{\rho} \tag{232}$$

or substituting the expression for $\Delta \rho$ from Eq. 231 into 232 and using the unit range vector $\underline{L}$ as defined in Eq. 45, the expression for $\Delta \dot{\rho}$ simplifies to

$$\Delta \dot{\rho} = \frac{1}{\rho} (\underline{\dot{\rho}} - \dot{\rho} \underline{L}) \cdot \Delta \underline{\rho} + \underline{L} \cdot \Delta \underline{\dot{\rho}} \tag{233}$$

Actually in the ODP the cartesian components of the station vector $\underline{R}$ are not used as parameters, but instead the geocentric radius R, latitude ϕ' and the longitude λ are subject to estimation. The reference coordinate system is the equatorial system of date so that the components of $\underline{R}$ are given by

$$X = R \cos \phi' \cos \theta \tag{234}$$

$$Y = R \cos \phi' \sin \theta \tag{235}$$

$$Z = R \sin \phi' \tag{236}$$

where the sidereal time θ is given by Eq. 44. In the station velocity vector $\dot{\underline{R}}$, the latitude, longitude and radius of the station are assumed invariant with time and thus only the angular velocity of rotation ω of the earth is involved as the time derivative of the sidereal time.

$$\omega \equiv \dot{\theta} \tag{237}$$

Therefore

$$\dot{X} = -\omega Y \tag{238}$$

$$\dot{Y} = \omega X \tag{239}$$

$$\dot{Z} = 0 \tag{240}$$

The variations in $\underline{R}$ and $\dot{\underline{R}}$ are therefore

$$\Delta X = \frac{X}{R} \Delta R - Z \cos \theta \Delta \phi' - Y \Delta \lambda \tag{241}$$

$$\Delta Y = \frac{Y}{R} \Delta R - Z \sin \theta \Delta \phi' + X \Delta \lambda \tag{242}$$

$$\Delta Z = \frac{Z}{R} \Delta R + R \cos \phi' \Delta \phi' \tag{243}$$

$$\Delta \dot{X} = -\omega \Delta Y \tag{244}$$

$$\Delta \dot{Y} = \omega \Delta Y \tag{245}$$

$$\Delta \dot{Z} = 0 \tag{246}$$

where from Eq. 44 $\Delta \theta = \Delta \lambda$. (247)

The variations in the angles are easily derived by a technique due to Herrick. Write the range vector in the form

$$\underline{\rho} = \rho \, \underline{L} \tag{248}$$

Then

$$\Delta \underline{\rho} = \rho \Delta \underline{L} + \underline{L} \Delta \rho \tag{249}$$

From Eq. 46

$$\Delta L_x = - \sin \delta \cos \alpha \, \Delta \delta - \cos \delta \sin \alpha \Delta \alpha \tag{250}$$

$$\Delta L_y = - \sin \delta \sin \alpha \, \Delta \delta + \cos \delta \cos \alpha \Delta \alpha \tag{251}$$

$$\Delta L_z = \cos \delta \Delta \delta \tag{252}$$

Define two new unit vectors in terms of the above linear coefficients.

$$\underline{A} = (- \sin \alpha \, , \cos \alpha \, , 0) \tag{253}$$

$$\underline{D} = (- \sin \delta \cos \alpha \, , - \sin \delta \sin \alpha \, , \cos \delta) \tag{254}$$

It is easy to show that $\underline{L}$, $\underline{A}$ and $\underline{D}$ form an orthonormal system. Also Eqs. 250,,251 and 252 can be written as a single vector equation

$$\Delta \underline{L} = \underline{A} \cos \delta \Delta \alpha + \underline{D} \, \Delta \delta \tag{255}$$

and Eq. 249 takes on the form

$$\Delta \underline{\rho} = \rho \, \underline{A} \cos \delta \, \Delta \alpha + \rho \, \underline{D} \, \Delta \delta + \underline{L} \, \Delta \rho \tag{256}$$

Take the scaler product of Eq. 256 with first $\underline{L}$, then $\underline{A}$ and finally $\underline{D}$ to obtain

$$\Delta \rho = \underline{L} \cdot \Delta \underline{\rho} \tag{257}$$

$$\rho \cos \delta \Delta \alpha = \underline{A} \cdot \Delta \underline{\rho} \tag{258}$$

$$\rho \, \Delta \delta = \underline{D} \cdot \Delta \underline{\rho} \tag{259}$$

The first of the resulting expressions Eq. 257 is identical with Eq. 231. However Eqs. 258 and 259 yield the variations in the right ascension α and declination δ. The variation in the hour angle H is obtained from the definitive expression Eq. 50

$$\Delta H = \Delta\lambda - \Delta\alpha \tag{259}$$

To obtain the variations in the azimuth σ and elevation angle γ it is necessary to write the range vector $\underline{\rho}$ in terms of the topocentric unit range vector $\underline{L}_h$ defined in section IIIB. First of all express the transformation equations, Eqs. 52, 53 and 54 in matrix notation

$$\underline{L} = A\underline{L}_h \tag{260}$$

where the rotation matrix A is a function of the geocentric station latitude ϕ' and the local sidereal time Θ. The elements of A are

$$a_{11} = \sin\phi' \cos\Theta \tag{261}$$

$$a_{12} = -\sin\Theta \tag{262}$$

$$a_{13} = \cos\phi' \cos\Theta \tag{263}$$

$$a_{21} = \sin\phi' \sin\Theta \tag{264}$$

$$a_{22} = \cos\Theta \tag{265}$$

$$a_{23} = \cos\phi' \sin\Theta \tag{266}$$

$$a_{31} = -\cos \phi' \tag{267}$$

$$a_{32} = 0 \tag{268}$$

$$a_{33} = \sin \phi' \tag{269}$$

The range vector in terms of $\underline{L}_h$ is from Eq. 248

$$\underline{\rho} = \rho \, A \, \underline{L}_h \tag{270}$$

and

$$\Delta \underline{\rho} = A\underline{L}_h \, \Delta \rho + \rho \Delta A \underline{L}_h + \rho A \Delta \underline{L}_h \tag{271}$$

The matrix ΔA can be expressed in the form

$$\Delta A = A_\phi \Delta \phi' + A_\theta \Delta \lambda \tag{272}$$

where the elements of A_ϕ and A_θ are obtained by differentiating Eqs. 261 to 269.

$$A_\phi = \begin{pmatrix} a_{13} & 0 & -a_{11} \\ a_{23} & 0 & -a_{21} \\ a_{33} & 0 & -a_{31} \end{pmatrix} \tag{273}$$

$$A_{\theta} = \begin{pmatrix} -a_{21} & -a_{22} & -a_{23} \\ a_{11} & a_{12} & a_{13} \\ 0 & 0 & 0 \end{pmatrix} \tag{274}$$

The variation $\Delta \underline{L}_h$ is expressed in a form analogous to $\Delta \underline{L}$ in Eq. 255.

$$\Delta \underline{L}_h = \underline{A}_h \cos\gamma \Delta\sigma + \underline{D}_h \Delta\gamma \tag{275}$$

where

$$\underline{A}_h = (\sin\sigma,\ \cos\sigma,\ 0) \tag{276}$$

and

$$\underline{D}_h = (\sin\gamma\cos\sigma,\ -\sin\gamma\sin\sigma,\ \cos\gamma) \tag{277}$$

Therefore Eq. 271 expressed in terms of variations in the observables and station coordinates is

$$\Delta \underline{\rho} = A\underline{L}_h \Delta\rho + \rho A\underline{A}_h \cos\gamma \Delta\sigma + \rho A\underline{D}_h \Delta\gamma$$

$$+ \rho A_{\phi} \underline{L}_h \Delta\phi' + \rho A_{\theta} \underline{L}_h \Delta\lambda \tag{278}$$

Again the vectors $\underline{L}_h$, $\underline{A}_h$ and $\underline{D}_h$ form an orthonormal system and thus $A\underline{L}_h$, $A\underline{A}_h$ and $A\underline{D}_h$ do also. Define these orthonormal vectors by

$$\underline{\tilde{L}} = A\underline{L}_h \tag{279}$$

$$\underline{\tilde{A}} = A\underline{A}_h \tag{280}$$

$$\underline{\tilde{D}} = A\underline{D}_h \tag{281}$$

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also

$$\tilde{\underline{L}}_{\phi} = A_{\phi}\underline{L}_{h} \tag{282}$$

$$\tilde{\underline{L}}_{\theta} = A_{\theta}\underline{L}_{h} \tag{283}$$

Then

$$\Delta \underline{\rho} = \tilde{\underline{L}}\Delta\rho + \rho\tilde{\underline{A}} \cos\gamma\Delta\sigma + \rho\, \tilde{\underline{D}}\Delta\gamma + \rho\, \tilde{\underline{L}}_{\phi}\Delta\phi' + \rho\, \tilde{\underline{L}}_{\theta}\Delta\lambda \tag{284}$$

and

$$\rho \cos \gamma\Delta\sigma = \tilde{\underline{A}}\cdot\Delta\underline{\rho} - \rho\, \tilde{\underline{A}}\cdot\tilde{\underline{L}}_{\phi}\Delta\phi' - \rho\tilde{\underline{A}}\cdot\tilde{\underline{L}}_{\theta}\Delta\lambda \tag{285}$$

and

$$\rho\Delta\gamma = \tilde{\underline{D}}\cdot\Delta\underline{\rho} - \rho\tilde{\underline{D}}\cdot\tilde{\underline{L}}_{\phi}\Delta\phi' - \rho\tilde{\underline{D}}\cdot\tilde{\underline{L}}_{\theta}\Delta\lambda \tag{286}$$

The vectors $\tilde{\underline{A}}$ and $\tilde{\underline{D}}$ in component form are

$$\tilde{A}_x = \sin\phi' \cos\theta \sin\sigma - \sin\theta \cos\sigma \tag{287}$$

$$\tilde{A}_y = \sin\phi' \sin\theta \sin\sigma + \cos\theta \cos\sigma \tag{288}$$

$$\tilde{A}_z = -\cos\phi' \sin\sigma \tag{289}$$

$$\tilde{D}_x = \sin\gamma \left[\sin\theta \sin\sigma + \sin\phi' \cos\theta \cos\sigma\right] + \cos\phi' \cos\theta \cos\gamma \tag{290}$$

$$\tilde{D}_y = \sin\gamma \left[\cos\sigma \sin\phi' \sin\theta - \sin\sigma \cos\theta\right] + \cos\phi' \sin\theta \cos\gamma \tag{291}$$

$$\tilde{D}_z = \sin \phi' \cos \gamma - \cos \phi' \cos \sigma \sin \gamma \tag{292}$$

while $\tilde{L}_\phi$ and $\tilde{L}_\theta$ are given by

$$\tilde{L}_{\phi x} = - \cos \theta \left[\cos \phi' \cos \gamma \cos \sigma + \sin \phi' \sin \gamma \right] \tag{293}$$

$$\tilde{L}_{\phi y} = - \sin \theta \left[\cos \phi' \cos \gamma \cos \sigma + \sin \phi' \sin \gamma \right] \tag{294}$$

$$\tilde{L}_{\phi z} = \cos \phi' \sin \gamma - \sin \phi' \cos \gamma \cos \sigma \tag{295}$$

$$\tilde{L}_{\theta x} = \sin \theta \left[\sin \phi' \cos \gamma \cos \sigma - \cos \phi' \sin \gamma \right] - \cos \theta \cos \gamma \sin \sigma \tag{296}$$

$$\tilde{L}_{\theta y} = - \cos \theta \left[\sin \phi' \cos \gamma \cos \sigma - \cos \phi' \sin \gamma \right] - \sin \theta \cos \gamma \sin \sigma \tag{297}$$

$$\tilde{L}_{\theta z} = 0 \tag{298}$$

C. Inversion of the State Transition Matrix

The formula Eq. 187 for the computation of the parameter sensitivity matrix V requires the inverse of the state transition matrix U. As the time t from the epoch t_o increases the matrix U becomes more and more ill-conditioned and a straight forward numerical inversion of U can result in a considerable loss in the significance of its inverse. Fortunately under certain applicable conditions the matrix can be inverted by inspection and it is the purpose of this section to define the necessary conditions under which this can be done.*

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* We wish to express our appreciation to J.B. McGuire for pointing out that this inversion by inspection is possible. Also the derivations, particularly the proof of Theorem 1., were largely suggested by some unpublished work of O.K. Smith.

Theorem 1.

Consider the system of differential equations

$$\frac{dU}{dt} = \theta(t)\ U;\ U(t_o) = I \tag{299}$$

where U and θ are 6 x 6 matrices and I is the unit matrix. Now partition U and θ into 3 x 3 submatrices as follows:

$$\theta = \left(\begin{array}{c|c} \theta_{11} & \theta_{12} \\ \hline \theta_{21} & \theta_{22} \end{array}\right)$$

$$U = \left(\begin{array}{c|c} u_{11} & u_{12} \\ \hline u_{21} & u_{22} \end{array}\right)$$

Then if the submatrices of θ obey the conditions that

$$\theta_{11} = -\ \theta_{22}^{T} \tag{300}$$

$$\theta_{12} = \theta_{12}^{T} \tag{301}$$

$$\theta_{21} = \theta_{21}^{T} \tag{302}$$

the inverse of the solution matrix u is given by

$$U^{-1} = \left(\begin{array}{c|c} u_{22}^{T} & -u_{12}^{T} \\ \hline -u_{21}^{T} & u_{11}^{T} \end{array}\right) \tag{303}$$

Proof

Introduce the matrix

$$E = \left(\begin{array}{c|c} 0 & e \\ \hline -e & 0 \end{array}\right) \tag{304}$$

where e is a 3 x 3 unit matrix.

Then:

$$E\,\Theta = \left(\begin{array}{c|c} \theta_{21} & \theta_{22} \\ \hline -\theta_{11} & -\theta_{12} \end{array}\right) \tag{305}$$

and

$$E\,\Theta^T = \left(\begin{array}{c|c} \theta_{21}^T & -\theta_{11}^T \\ \hline \theta_{22}^T & -\theta_{12}^T \end{array}\right) \tag{306}$$

by the conditions imposed on Θ

$$(E\Theta) = (E\Theta)^T \tag{307}$$

$$(U^T E\Theta U) = (U^T E\Theta U)^T \tag{308}$$

or multiplying the differential equation by $U^T E$ and equating the resulting matrix with its transpose yields

$$U^T E \ \frac{dU}{dt} = \left(\frac{dU}{dt}\right)^T \ E^T U \tag{309}$$

because $E^T = -E$ it follows that

$$U^T E \ \frac{dU}{dt} + \left(\frac{dU}{dt}\right)^T \ EU \equiv 0 \tag{310}$$

We recognize this expression as the derivative of $(U^T EU)$. Thus $(U^T EU)$ is a constant given by

$$U^T EU = U^T(t_o)\ EU(t_o) = E \tag{311}$$

Because $E^T E = I$

$$E^T U^T E U = I$$

and

$$U^{-1} = E^T\ U^T E = -E\ U^T E$$

Performing the indicated multiplications proves the theorem.

Space Fixed Cartesian Coordinates

We will now show that cartesian position and velocity coordinates satisfy the conditions of Theorem 1. for certain dynamical systems, in particular for motion in a field describable by a potential function. The following theorem is sufficient to establish this result.

Theorem 2.

Let $(x_1, x_2, x_3, \dot{x}_1, \dot{x}_2, \dot{x}_3,)$ be a set of independent coordinates such that for a given dynamical system the equations of motion are of the form:

$$\dot{x}^{\cdot}_r = f_r(x_1, x_2, x_3, t) \qquad r = 1, 2, 3 \tag{312}$$

Then if

$$\frac{\partial \dot{x}^{\cdot}_r}{\partial x_s} = \frac{\partial \dot{x}^{\cdot}_s}{\partial x_r} \qquad r, s = 1, 2, 3 \tag{313}$$

the inverse of the matrix

$$U = \left(\begin{array}{c|c} \left(\frac{\partial x_r}{\partial x_{so}}\right) & \left(\frac{\partial x_r}{\partial \dot{x}_{so}}\right) \\ \hline \left(\frac{\partial \dot{x}_r}{\partial x_{so}}\right) & \left(\frac{\partial \dot{x}_r}{\partial \dot{x}_{so}}\right) \end{array}\right) \tag{314}$$

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Is given by

$$U^{-1} = \left(\begin{array}{c|c} \left(\dfrac{\partial \dot{x}_r}{\partial \dot{x}_{so}}\right)^T & \left(\dfrac{\partial x_r}{\partial \dot{x}_{so}}\right)^T \\ \hline -\left(\dfrac{\partial \dot{x}_r}{\partial x_{so}}\right)^T & \left(\dfrac{\partial x_r}{\partial x_{so}}\right)^T \end{array}\right) \tag{315}$$

The symbol (a_{rs}) represents the matrix with elements a_{rs} and $(x_{10}, x_{20}, x_{30}, \dot{x}_{10}, \dot{x}_{20}, \dot{x}_{30})$ is the value of the coordinates at time t_o.

Proof

All that is required is to show that conditions (300), (301) and (302) of Theorem 1., are satisfied. Because of independence the θ_{11} and θ_{12} submatrices of the θ matrix are

$$\theta_{11} = \left\{\frac{\partial \dot{x}_r}{\partial x_s}\right\} = 0 \tag{316}$$

$$\theta_{12} = \left\{\frac{\partial \dot{x}_r}{\partial \dot{x}_s}\right\} = I \tag{317}$$

Also from (6)

$$\theta_{22} = \left\{\frac{\partial \dot{x}^{\cdot}_r}{\partial \dot{x}_s}\right\} = 0 \tag{318}$$

Thus

$$\theta_{11} = - \theta_{22}^T \tag{319}$$

$$\theta_{12} = \theta_{22}^T \tag{320}$$

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which satisfy conditions (300) and (301). Condition (302) is satisfied by Eq. (313) because

$$\theta_{21} = \left(\frac{\partial \dot{x}^{\cdot}_r}{\partial x_s}\right) = \left(\frac{\partial \dot{x}^{\cdot}_s}{\partial x_r}\right) = \theta_{21}^T \tag{321}$$

Suppose now that the motion can be described by a potential function Φ so that in cartesian coordinates

$$\dot{x}^{\cdot}_r = \frac{\partial \Phi}{\partial x_r} \tag{322}$$

Then

$$\frac{\partial \dot{x}^{\cdot}_r}{\partial x_s} = \frac{\partial^2 \Phi}{\partial x_s \, \partial x_r} \tag{323}$$

and

$$\frac{\partial \dot{x}^{\cdot}_s}{\partial x_r} = \frac{\partial^2 \Phi}{\partial x_r \, \partial x_s} \tag{324}$$

Therefore at all points where $\frac{\partial \dot{x}^{\cdot}_r}{\partial x_s}$ and $\frac{\partial \dot{x}^{\cdot}_s}{\partial x_r}$ are continuous, condition (313) of Theorem 2. holds. Also if condition (312) is satisfied the matrix U can be inverted as indicated. This is precisely the situation for the motion of a space probe not subject to velocity dependent accelerations.

JDA/cw:bmc:llb

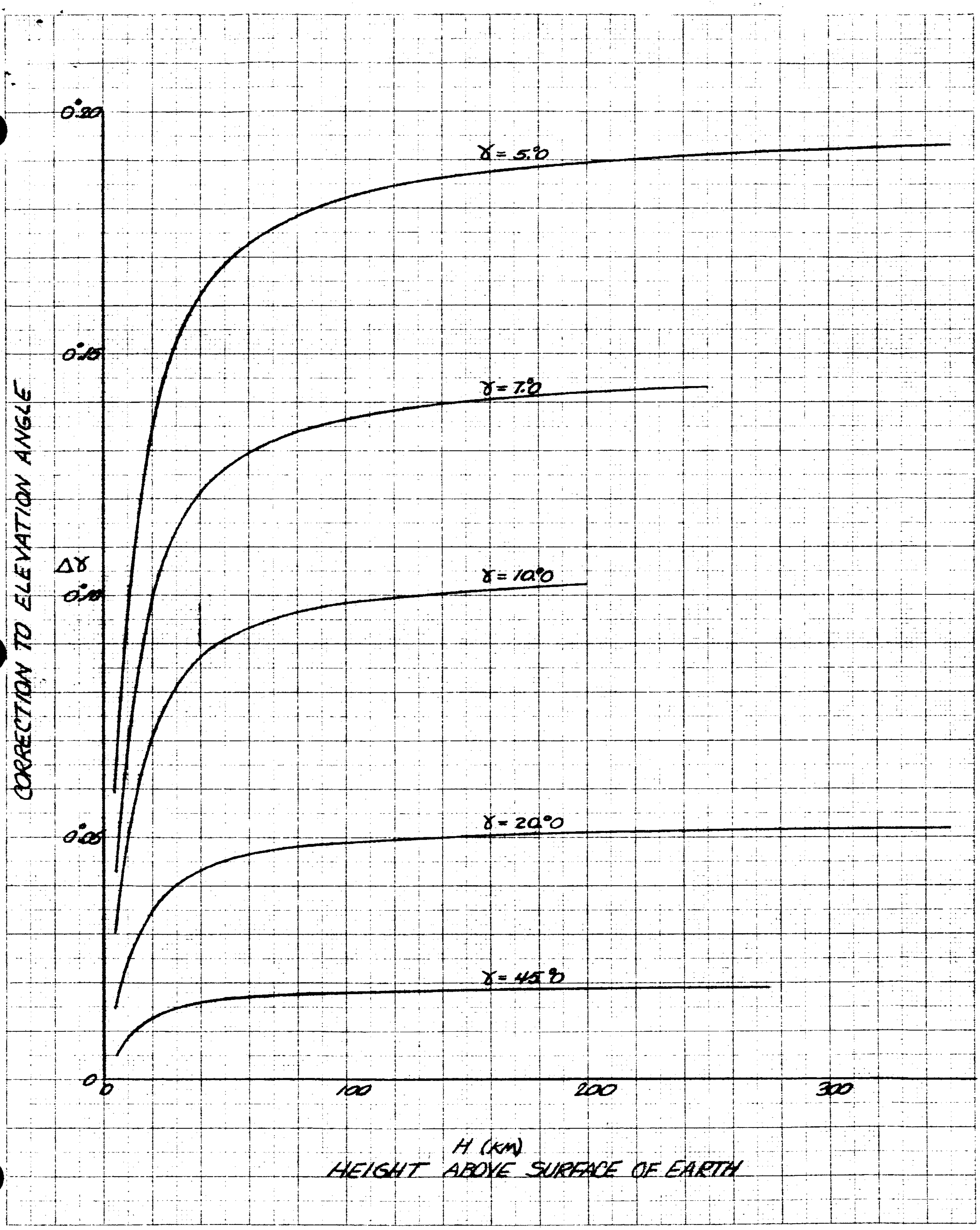
0.20
0.15
0.10
0.05
0
Δγ
CORRECTION TO ELEVATION ANGLE
γ = 5.0°
γ = 7.0°
γ = 10.0°
γ = 20.0°
γ = 45.0°
0
100
200
300
H (KM)
HEIGHT ABOVE SURFACE OF EARTH

FIG. 12

[redacted]

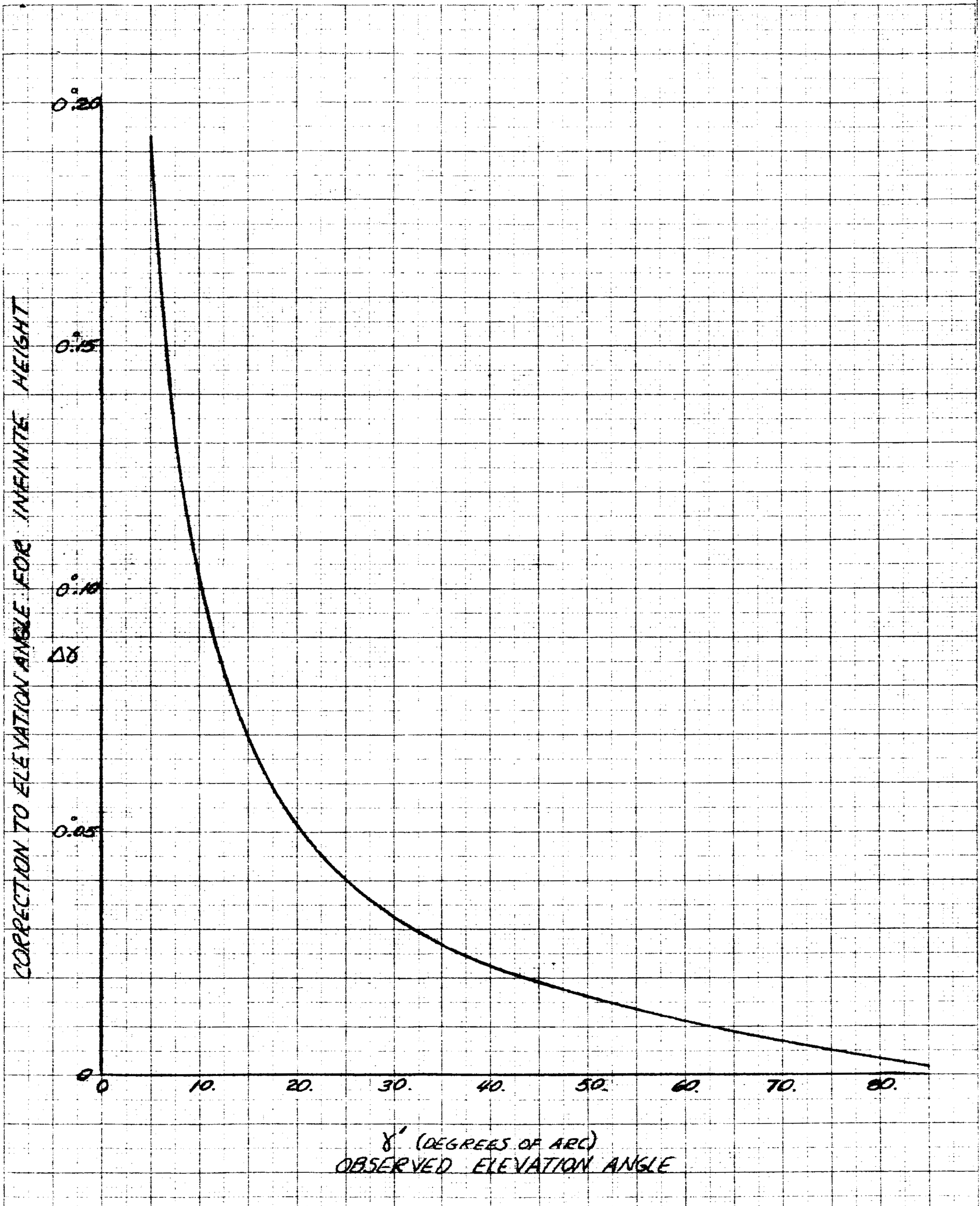

FIG. 13

[redacted]

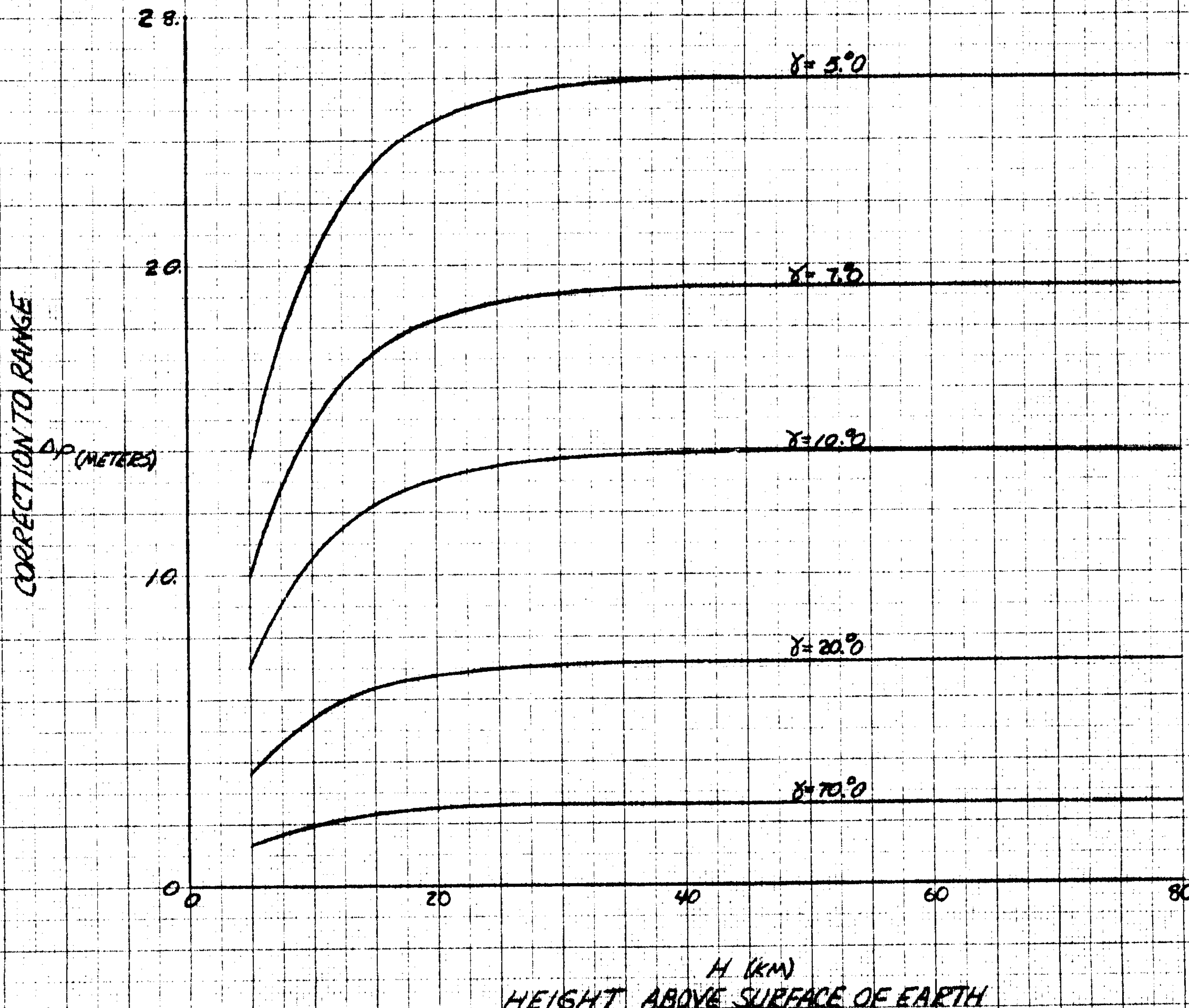
2 8.
2 0.
1 0.
0
CORRECTION TO RANGE
Δρ (METERS)
γ = 5.°0
γ = 7.°5
γ = 10.°0
γ = 20.°0
γ = 70.°0
0
20
40
60
80
H (KM)
HEIGHT ABOVE SURFACE OF EARTH

FIG. 14

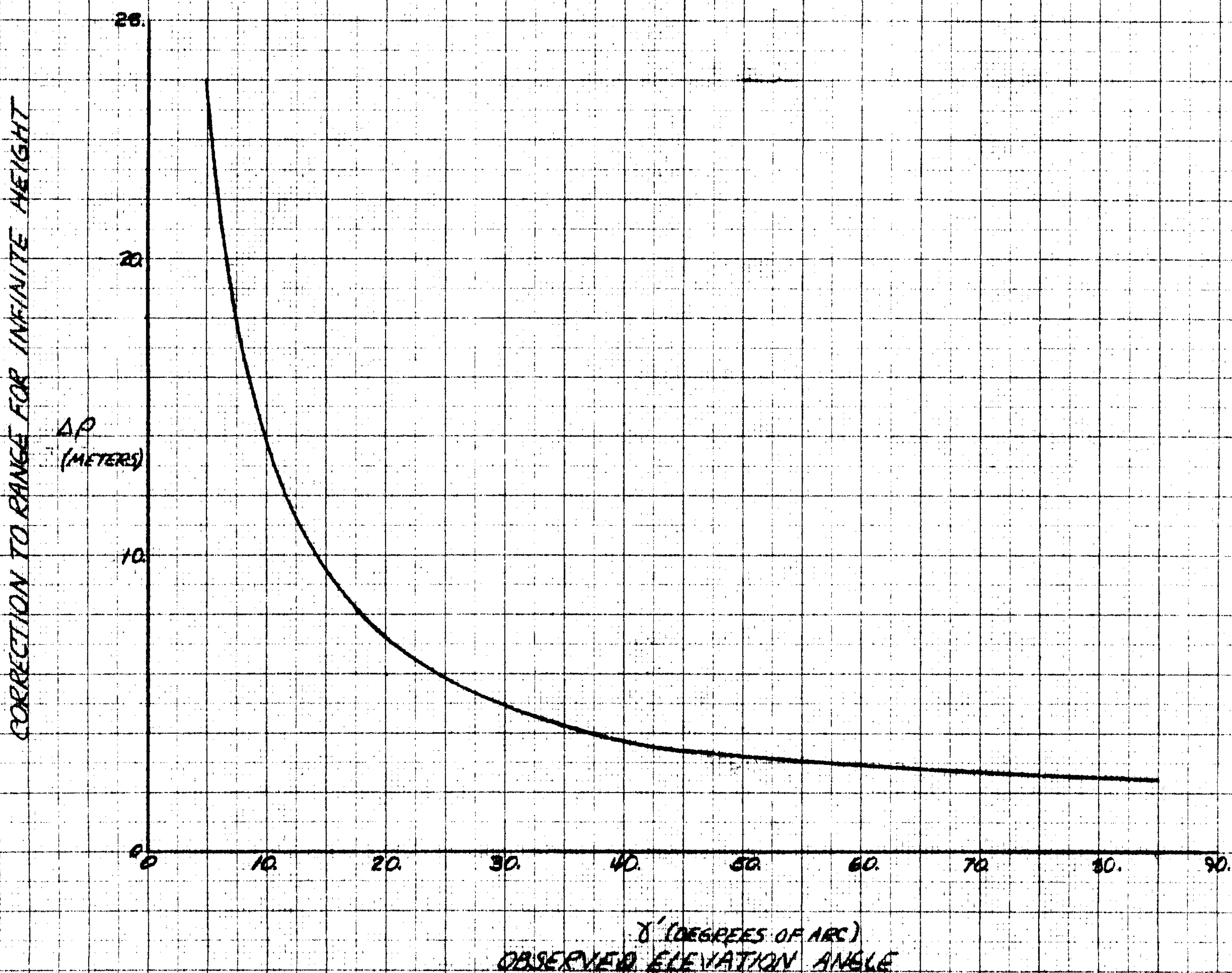
CORRECTION TO RANGE FOR INFINITE HEIGHT
$\Delta\rho$ (METERS)
0
10.
20.
26.
0
10.
20.
30.
40.
50.
60.
70.
80.
90.
γ' (DEGREES OF ARC)
OBSERVED ELEVATION ANGLE

FIG. 15

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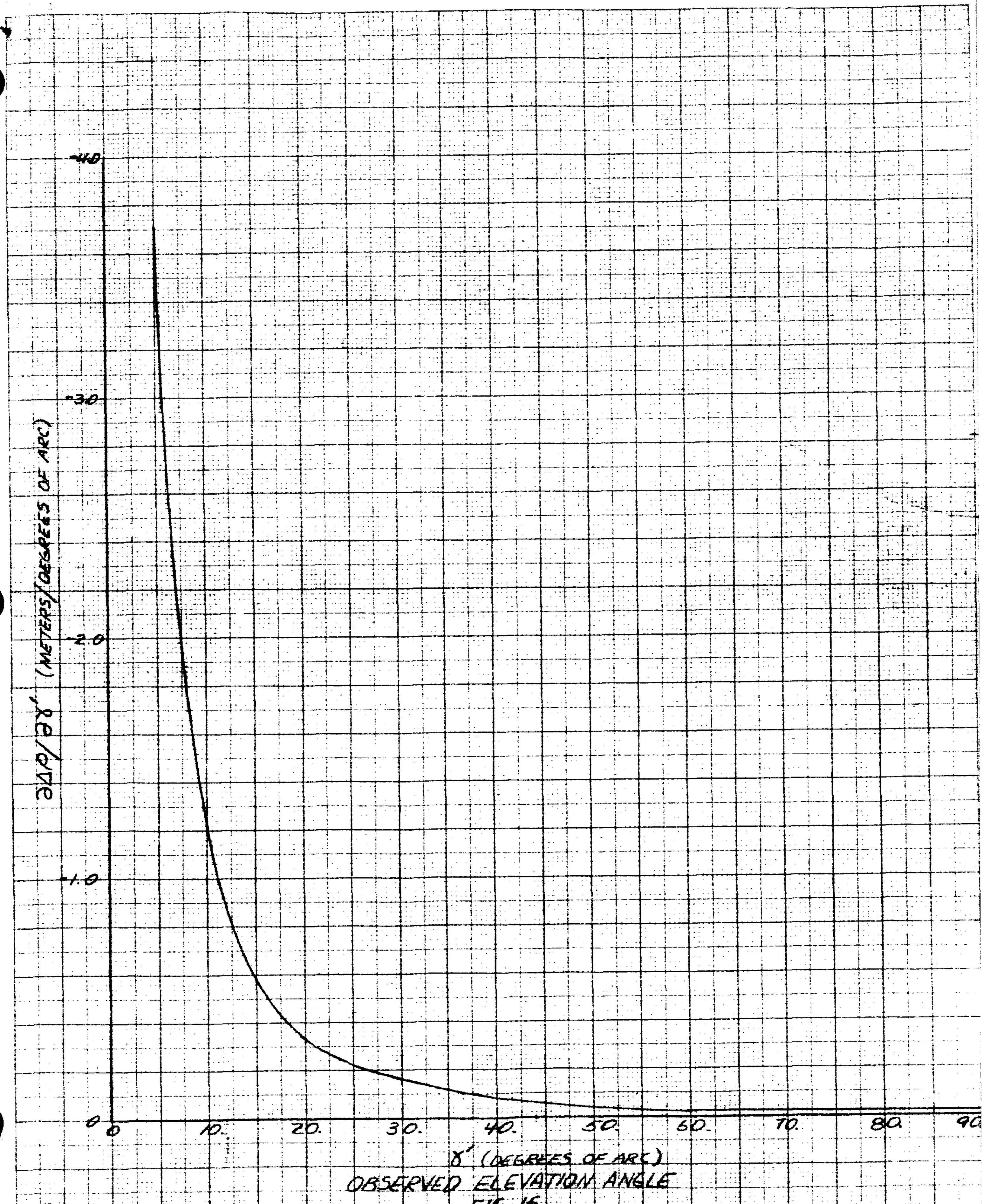
4.0
3.0
2.0
1.0
0
$\partial\Delta\rho/\partial\gamma'$ (METERS/DEGREES OF ARC)
0
10.
20.
30.
40.
50.
60.
70.
80.
90
γ' (DEGREES OF ARC)
OBSERVED ELEVATION ANGLE
FIG. 16